THE UNIVERSITY OF ALBERTA

VIBRATION OF THIN-WALLED OPEN SECTION BEAMS
SUBJECTED TO AXIAL LOAD

by

STANLEY ROY DIDRIKSON, B. Sc. (Alberta)

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MECHANICAL ENGINEERING

EDMONTON, ALBERTA

April 1964

ABSTRACT

The effect of axial load on the vibration frequency of a thin-walled open section beam is investigated both theoretically and experimentally. A method of solution of the governing differential equations for various boundary conditions using a digital computer is presented. Solutions are shown for beams of both symmetrical and unsymmetrical angle cross section with built-in ends and these are compared with experimental results. A practical application of this theory is discussed and areas where further work is required are indicated.

ACKNOWLEDGEMENTS

The author wishes to extend his appreciation to Dr. J. S. Kennedy for supervising this thesis. Thanks must also go to all the members of the Mechanical Engineering Department who have been of assistance in numerous ways, and particularly to Hans Golls and Ray Marak who were largely responsible for building and instrumenting the experimental apparatus.

The help offered by the members of the University of Alberta Computing Department has been sincerely appreciated. Particular mention must go to David Simpson, Harry Maximchuk, and Cordel Rolfson, whose computer programs were used.

The author would also like to acknowledge his gratitude to the members of the Electrical Engineering Department who provided assistance.

Thanks are extended to Mrs. B. Lloyd, who typed the manuscript with much care and patience, and to Charles Didrikson, who skillfully made the drawings and graphs.

The author was given leave by the R.C.A.F. to do his graduate work. He would like to express his appreciation to Dr. G. Ford and Wing Commander B. Riedel for their considerable efforts in making this possible.

TABLE OF CONTENTS

CHAPTER		PAGE
I.	INTRODUCTION	1
	1.1 Primary Considerations	1
	1.2 Historical Review	3
	1.3 Aim of the Thesis	5
II.	THEORETICAL CONSIDERATIONS	6
	2.1 Nonuniform Torsion	6
	2.2 Derivation of the Differential Equations	9
	2.3 Application of Method of Separation of Variables	16
	2.4 Boundary Conditions	17
	2.5 Solution for Simply Supported Ends . .	19
	2.6 General Solution	21
	2.7 Unequal Leg Angle Solution	30
	2.8 Equal Leg Angle Solution	30
III.	EXPERIMENTAL APPARATUS AND PROCEDURE	34
	3.1 Test Specimens	34
	3.2 Apparatus	35
	3.3 Test Procedure	39
IV.	RESULTS AND DISCUSSION	47
	4.1 Simply Supported Ends	47
	4.2 Theoretical Results, Fixed Ends	48
	4.3 Experimental Results	51

CHAPTER PAGE

4.4 Possible Practical Application 54

V. CONCLUSIONS AND RECOMMENDATIONS 74

5.1 Conclusions 74

5.2 Recommendations 75

BIBLIOGRAPHY 77

APPENDIX 79

Fortran Source Programs 79

LIST OF TABLES

TABLE		PAGE
3.1	Test Specimens	41
3.2	Strain Gage Details	41
3.3	Strain Gage Locations	41
3.4	Frequency Response Check of Sanborn Recorder. .	41
4.1	Effect of Axial Load on Frequency, Simple Supports	56
4.2	Effect of Axial Load on Frequency, Beam #1, Theoretical Results	59
4.3	Effect of Axial Load on Frequency, Beam #1, Experimental Results	60
4.4	Effect of Variation in Elastic Constants . . .	63
4.5	Effect of Axial Load on Frequency, Beam #2, Theoretical Results	65
4.6	Effect of Axial Load on Frequency, Beam #2, Experimental Results	66
4.7	Effect of Axial Load on Frequency, Unsymmetrical Beams, Theoretical Results . .	69
4.8	Effect of Axial Load on Frequency, Unsymmetrical Beams, Experimental Results . .	70
4.9	Comparison of Actual Axial Loads with Values Obtained Using Frequency Measurements	73

LIST OF FIGURES

FIGURE		PAGE
2.1	Orientation of Coordinate System	10
2.2	Orientation of Principal Axes	10
2.3	Simple Strut Under Axial Thrust	13
2.4	Member of Arbitrary Unsymmetrical Open Cross Section Subjected to Axial Thrust . .	13
3.1	Schematic View of Experimental Apparatus . . .	42
3.2	Experimental Apparatus	43
3.3	General View of Apparatus	43
3.4	Slide and Yoke Assembly, Hydraulic Ram and Valve	44
3.5	Slide Assembly	44
3.6	Vibration Recording Equipment	45
3.7	Strain Indicator and Switching Units	45
3.8	Details of Strain Gage Attachment	46
3.9	Fixed End Clamps	46
4.1	Effect of Axial Load on Vibration Frequency, Beam #1, Simple Supports	57
4.2	Axial Load - Square of Frequency Relation, Beam #1, Simple Supports	58
4.3	Effect of Axial Load on Vibration Frequency, Beam #1, Fixed Ends	61
4.4	Axial Load - Square of Frequency Relation, Beam #1, Fixed Ends	62
4.5	Effect of Variation in E and G	64

FIGURE | | PAGE

4.6 Effect of Axial Load on Vibration Frequency, Beam #2, Fixed Ends 67

4.7 Axial Load - Square of Frequency Relation, Beam #2, Fixed Ends 68

4.8 Effect of Axial Load on Vibration Frequency, Unsymmetrical Beams, Fixed Ends 71

4.9 Axial Load - Square of Frequency Relation, Unsymmetrical Beams, Fixed Ends 72

NOTATION

c	Torsion constant
c_w	Warping constant
c_y, c_z	Coordinates of centroid
e	Eccentricity, distance from shear center to centroid for cross sections with one degree of symmetry
i	Integer, indicative of a given term in a series
l	Length of beam
m	Mass density of material in beam
n	Integer, indicative of deflection mode
p	Total length of middle line of cross section
p_n	Natural frequency of vibration
p_y, p_z	Uncoupled flexural natural frequencies
p_ϕ	Uncoupled torsional natural frequency
r	Roots of characteristic equation
r_t	Distance from the tangent at the point under consideration on the middle line to the axis of rotation
s	Distance along the middle line of the cross section
t	Time, thickness
u	Warping displacement of a cross section
v, w	Deflection of the shear center in the y- and z-directions, respectively
w_s, $\bar{w}_s$	Warping function, mean value of warping function
w_t	Intensity of distributed torque acting along the shear center axis

w_y, w_z	Intensities of distributed torque acting along the shear center axis
x, y, z	Rectangular coordinates
A	Cross sectional area
D	Operator, where $D = \frac{d}{dx}$
E	Modulus of Elasticity
G	Modulus of Rigidity
I_o, I_p	Polar moments of inertia about centroid and shear center, respectively
I_η, I_ζ	Principal centroidal moments of inertia
M_t	Torque
M_y, M_z	Moments about the y- and z-axes of the cross section, respectively, produced by lateral loads
R	Beam radius of curvature
S	Axial thrust
T(t)	Deflection of shear center as a function only of time
V(x),W(x)	Deflection of shear center in y- and z-directions, respectively, as functions only of distance along beam
α	Small angle
γ	Function, where $\gamma_i = \psi_i/\chi_i$
ϵ	Strain
θ	Angle of twist per unit length
ξ, η, ζ	Rectangular coordinates
ρ	Roots of characteristic equation $\rho = r^2$
σ	Normal stress

ϕ Angle of twist

χ Function, where

$$\chi_i,\ \chi_{i+1} = -\frac{\left[\frac{EI_\eta}{mA} r_i^4 + \frac{S}{mA} r_i^2 - p_n^2\right]}{c_y\left[\frac{S}{mA} r_i^2 - p_n^2\right]}$$

ψ Function, where

$$\psi_i,\ \psi_{i+1} = \frac{\left[\frac{EI_\zeta}{mA} r_i^4 + \frac{S}{mA} r_i^2 - p_n^2\right]}{c_z\left[\frac{S}{mA} r_i^2 - p_n^2\right]}$$

$\Phi(x)$ Angle of rotation of cross section as a function only of distance along beam

CHAPTER I

INTRODUCTION

1.1 Primary Considerations

This thesis reports on investigation into the effect of axial end loads on the vibration frequencies in thin-walled open section beams. Such beams are extensively used in modern construction, especially in aircraft, because of their strength and lightness. A study of a vibration problem of members with such applications is therefore of practical interest.

As this subject involves torsional as well as flexural vibrations, it will be necessary to first discuss some aspects of the theory of torsion. In general, torsion of a beam produces warping, which is the displacement in the longitudinal direction of the beam of a point on the cross section which originally was in a plane at right angles to the longitudinal axes; that is, it is the distortion of a plane cross section out of its plane. With thin-walled beams it is necessary to distinguish between pure or uniform torsion and nonuniform torsion. Pure torsion takes place if all the cross sections are free to warp and if the torque is applied only at the ends of the beam. The warping is then constant along the beam and takes place without any axial strain of the longitudinal fibers.

Nonuniform torsion occurs if torque is applied to a beam in which some of the cross sections are restrained from warping, or if the torque varies along the beam. Then warping varies along the beam so that the torsional shear stress is accompanied by tension or compression of the longitudinal fibers. The rate of change of the angle of twist also varies along the length of the beam.

There will be a coupling of the torsional and flexural vibrations when the longitudinal axis which passes through the mass center of the cross section is not collinear with the longitudinal axis about which the beam tends to twist under the influence of an applied torsional couple. The latter axis, the axis of twist, by definition passes through the shear center of the cross section, and may also be defined by the property that it is the only axis along which a transverse load applied to the beam will produce flexure without twisting.

When a beam in which these axes do not coincide is caused to vibrate in the transverse direction, torsional oscillations about the longitudinal axis tend to be superimposed upon the lateral motion because the resultant of the transverse shearing forces on any element of length acts through the shear center of the cross section whereas the inertia forces act through the mass center. A couple is thus produced which naturally introduces torsional

oscillations. The beam, then, behaves as a coupled elastic system with normal modes of vibration involving simultaneous displacements in flexure and torsion. It follows that the natural frequencies of such vibrations differ from the uncoupled frequencies, that is the frequencies which can be computed assuming pure flexural and pure torsional vibration.

If the beam cross section has an axis of symmetry, both the centroid and the shear center will be on that axis so that, for a homogeneous material, flexural vibration parallel to an axis of symmetry will not introduce a twisting moment and will not be coupled with a torsional vibration. Thus a beam of equal leg angle cross section will have one uncoupled flexural and two coupled flexural-torsional fundamental modes of vibration whereas an unequal leg angle, which is completely unsymmetrical, has three coupled fundamental modes. The former case is known as double coupling and the latter as triple coupling.

1.2 Historical Review

The problem of vibration of thin-walled bars was discussed very generally by Vlasov (1, 2) in 1940. He derived differential equations for coupled vibrations of thin-walled beams of arbitrary cross section with eccentric axial end loads and showed that they could be solved for the particular case of boundary conditions corresponding to

the case where deflections only are restrained at the ends of the beam, which will be called "simple supports." Vlasov found the solution for this problem by applying the method of separation of variables and taking the deflected form as that of a sine wave function.

Independently Garland (3), 1940, investigated flexural-torsional vibrations of cantilever beams using the Rayleigh-Ritz method and verified his results experimentally. Timoshenko (4) also considered the problem as applied to a channel with simple supports and Frederhoffer (5) solved a general case for asymmetrical cross sections. Karyakin (6) presented a solution for an arbitrary cross section using energy methods.

Gere (7) extensively investigated torsional vibrations of beams of thin-walled open cross section and the effect of warping of the cross section in 1954. He also considered the effect of several kinds of end conditions. Gere and Lin (8), in 1957, gave a thorough treatment to the problem of coupled vibrations of thin-walled beams. They solved the differential equations for both double and triple coupling for thin-walled beams of arbitrary cross section for a variety of end conditions with the aid of a digital computer. They also discussed a solution by the Rayleigh-Ritz method. They did not, however, consider the effect of axial end loads.

1.3 Aim of the Thesis

The historical review shows that the subject of coupled vibrations of thin-walled beams of open cross section has been rather fully explored, both by direct solution of the differential equations and by the Rayleigh-Ritz method. Also, the effect of axial end loads on the vibration frequency has been generally discussed for the particular case of simple supports.

It is the purpose of this thesis to show that the effect of end loads on the vibration frequency of a beam of thin-walled open cross section can be obtained for end conditions other than simple supports by the direct solution of the differential equations with the aid of a digital computer. This is done by developing the general solution and then giving specific examples of double and triple coupling of beams of angle cross section with fixed end supports. The solutions are compared with experimental results.

CHAPTER II

THEORETICAL CONSIDERATIONS

In the following pages the differential equations for the vibration of a beam of thin-walled open cross section with axial end loads will be derived and solutions obtained. As nonuniform torsion is assumed in this problem, a short discussion of the subject will be given first. The derivation for the general problem of triple coupling follows. This derivation is similar to that presented by Gere and Lin (9) but considers the effect of axial end loads. It will be shown that the partial differential equations obtained can be reduced to ordinary differential equations by the method of separation of variables. For the sake of completeness the solution for the boundary conditions corresponding to simple supports will be given. The method of solution for any boundary conditions, employing a digital computer, will then be shown. Finally, examples of the solution for the particular cases of unequal and equal leg angles will be given.

2.1 Nonuniform Torsion

For pure torsion the torque M_t is given by

$$M_t = Gc\,\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad 2.1$$

The torsion constant c, for a beam of thin-walled open

section may be taken as

$$c = \frac{1}{3}\sum_i p_i t_i^3$$

Warping displacements u of a cross section are given by (10, 11)

$$u = \theta(\bar{w}_s - w_s)$$

where w_s is called the warping function and $\bar{w}_s$ is the average value of w_s. If the middle line of the cross section is defined as the intersection of a transverse plane through the beam with the surface midway between the two faces of the member, these functions can be written

$$w_s = \int_o^s r_t d\theta \quad , \qquad \bar{w}_s = \frac{1}{p}\int_o^p w_s ds$$

where s is distance along the middle line of the cross section to the point under consideration,

r_t is perpendicular distance from the tangent at the point under consideration on the middle line to the axis of rotation, and

p is the total length of the center line of the cross section.

For nonuniform torsion, the constant angle of twist per unit length θ is replaced by the variable rate of change of the angle of twist $\frac{d\phi}{dx}$. Then

$$u = (\bar{w}_s - w_s)\frac{d\phi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2.2$$

Since $\frac{d\phi}{dx}$ varies along the length of the beam, adjacent cross

sections will not be warped equally and there will be axial strain ϵ_x of the longitudinal fibers of the beam. As w_s and $\bar{w}_s$ are independent of x

$$\epsilon_x = \frac{\partial u}{\partial x} = (\bar{w}_s - w_s)\frac{d^2\phi}{dx^2}$$

It is assumed that there is no lateral pressure between the longitudinal fibers. Then the normal stresses produced during nonuniform torsion are given by Hooke's law

$$\sigma_x = E\epsilon_x = E(\bar{w}_s - w_s)\frac{d^2\phi}{dx^2} \qquad \text{.............} \; 2.3$$

The normal stresses σ_x produce shearing stresses which contribute a second term to the torque equation. For nonuniform torsion

$$M_t = Gc\frac{d\phi}{dx} - Ec_w\frac{d^3\phi}{dx^3} \qquad \text{..................} \; 2.4$$

where c_w is called the warping function and is defined by

$$c_w = \int_0^p (\bar{w}_s - w_s)^2\, t\, ds \qquad \text{..................} \; 2.5$$

It should be noted that for cross sectional shapes consisting of thin rectangular elements which intersect at a common point, the warping function, and thus the warping displacement as well as the warping constant, are all zero. This follows from the fact that the shear center is always at the common point of intersection of the rectangular elements for the cross sections mentioned. The distance r_t from the tangent at any point on the middle line of the cross section to the center of twist is then always zero.

Thus the warping function vanishes and there is no warping of the middle line during torsion. This fact applies, in particular, to cross sections of angle shape.

2.2. Derivation of the Differential Equations

Consider a beam of unsymmetrical cross section such as the one shown in Figures 2.1 and 2.2 subject to axial loads. The centroid and shear center are represented by points C and O, respectively. The y- and z-axes are through the shear center and parallel to the principal centroidal axes η and ζ . Figure 2.2 shows the beam in its equilibrium position. Bending vibrations in the y- and z-directions are measured by the deflections v, w of the shear center. These are coupled with torsional vibrations measured by the angle of rotation ϕ. The other symbols are as defined under "Notation."

It is assumed that the deflections are small and that ϕ is a small angle, that bending is about the centroidal axes and twisting about the longitudinal axis through the shear center of the section. The walls of the beam are thin, and the beam is long and slender, of constant cross section along its length and composed of a homogeneous material. The axial thrust S acts along the straight line joining the centroids of end cross sections, that is along a line parallel to the x-axis.

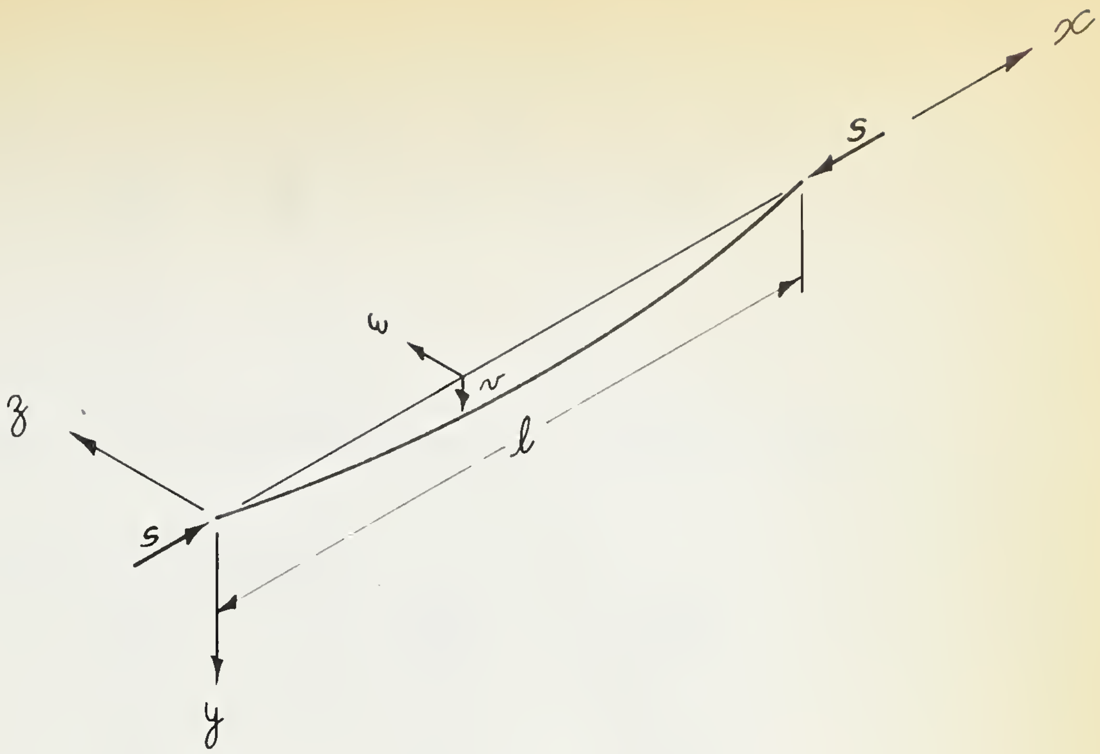


Fig. 2.1 ORIENTATION OF COORDINATE SYSTEM.

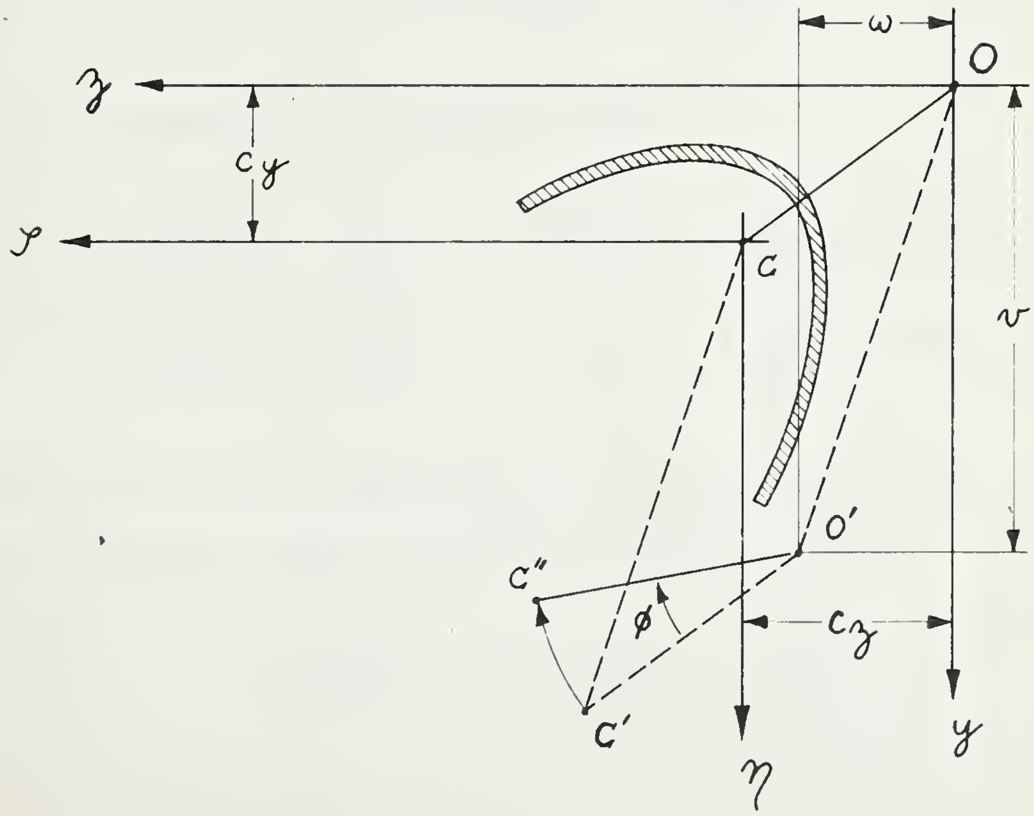


Fig. 2.2 ORIENTATION OF PRINCIPLE AXES FOR THIN-WALLED BEAM OF ARBITRARY CROSS-SECTION.

The equations of bending are

$$EI_\zeta \frac{d^2v}{dx^2} = -M_z - S(v - \phi c_z)$$

$$EI_\eta \frac{d^2w}{dx^2} = -M_y - S(w + \phi c_y)$$

where M is the moment produced by the lateral load.

Differentiating twice gives

$$EI_\zeta \frac{d^4v}{dx^4} = W_y - S(\frac{d^2v}{dx^2} - c_z \frac{d^2\phi}{dx^2})$$

$$EI_\eta \frac{d^4w}{dx^4} = W_z - S(\frac{d^2w}{dx^2} + c_y \frac{d^2\phi}{dx^2})$$

The inertia forces of translation are

$$W_y = -m\ A \frac{\partial^2}{\partial t^2} (v - c_z\phi)$$

$$W_z = -m\ A \frac{\partial^2}{\partial t^2} (w + c_y\phi)$$

By d'Alembert's principle, then

$$EI_\zeta \frac{\partial^4 v}{\partial x^4} + S\frac{\partial^2 v}{\partial x^2} + mA\frac{\partial^2 v}{\partial t^2} - Sc_z\frac{\partial^2\phi}{\partial x^2} - mAc_z\frac{\partial^2\phi}{\partial t^2} = 0 \qquad 2.6$$

$$EI_\eta \frac{\partial^4 w}{\partial x^4} + S\frac{\partial^2 w}{\partial x^2} + mA\frac{\partial^2 w}{\partial t^2} + Sc_y\frac{\partial^2\phi}{\partial x^2} + mAc_y\frac{\partial^2\phi}{\partial t^2} = 0 \qquad 2.7$$

Differentiating equation 2.4 gives the equation of nonuniform torsion under static torque:

$$Gc \frac{d^2\phi}{dx^2} - Ec_w \frac{d^4\phi}{dx^4} = -w_t$$

The total inertia torque is

$$(w_t)_I = -mI_p \frac{\partial^2 \phi}{\partial t^2} + mA \frac{\partial^2}{\partial t^2} (v - \phi c_z) c_z - mA \frac{\partial^2}{\partial t^2} (w + \phi c_y) c_y$$

$$(w_t)_I = -mI_o \frac{\partial^2 \phi}{\partial t^2} + mA \left(c_z \frac{\partial^2 v}{\partial t^2} - c_y \frac{\partial^2 w}{\partial t^2}\right)$$

It is now necessary to find the torque produced by the axial thrust S. This is done following a derivation given by Timoshenko and Gere (12). Consider a simple strut under axial thrust as shown in Figure 2.3

$$\alpha R = dx \qquad \frac{1}{R} \doteq -\frac{d^2 v}{dx^2}$$

$$\therefore \quad \alpha = -\frac{d^2 v}{dx^2} dx$$

Thus the axial load on the ends of the beam gives the effect of a transverse load

$$- S \frac{d^2 v}{dx^2} dx$$

on every element dx. In other words, the axial thrust has the effect of a transverse load of intensity

$$- S \frac{d^2 v}{dx^2}$$

Now consider a member or arbitrary unsymmetrical open cross section subjected to axial thrust as shown in Figure 2.4. Take a strip of cross sectional area tds defined by the coordinates y, z in the plane of the cross

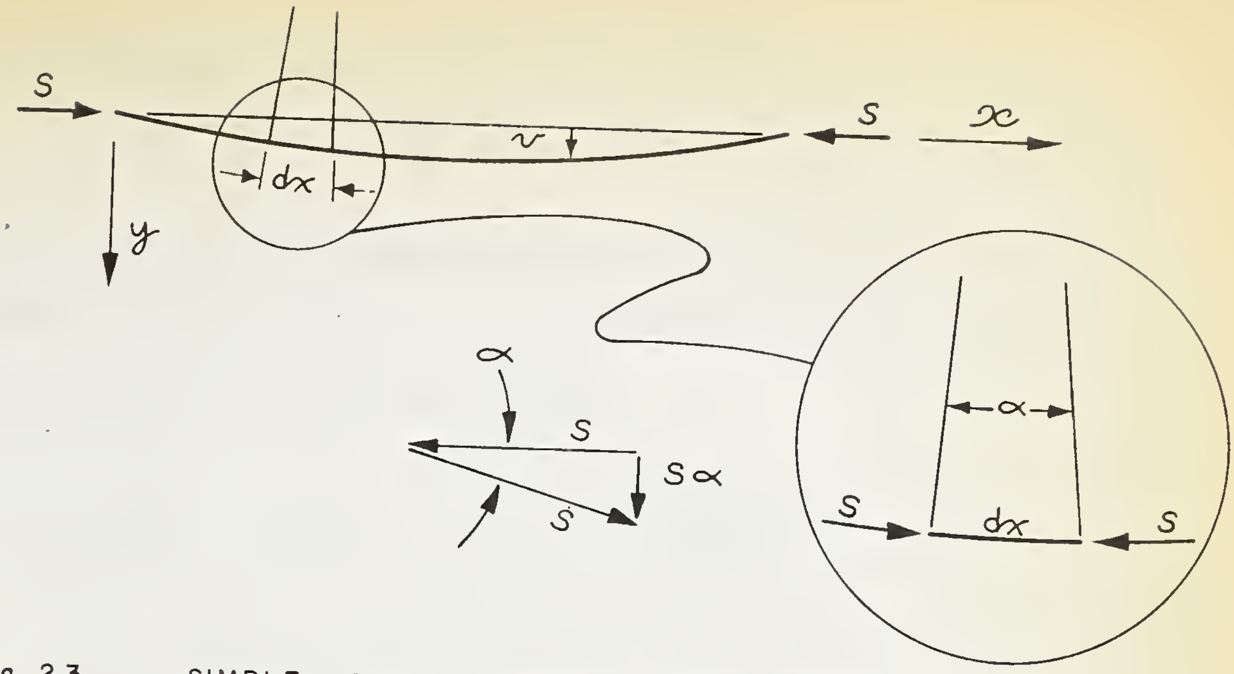


Fig. 2.3 SIMPLE STRUT UNDER AXIAL THRUST.

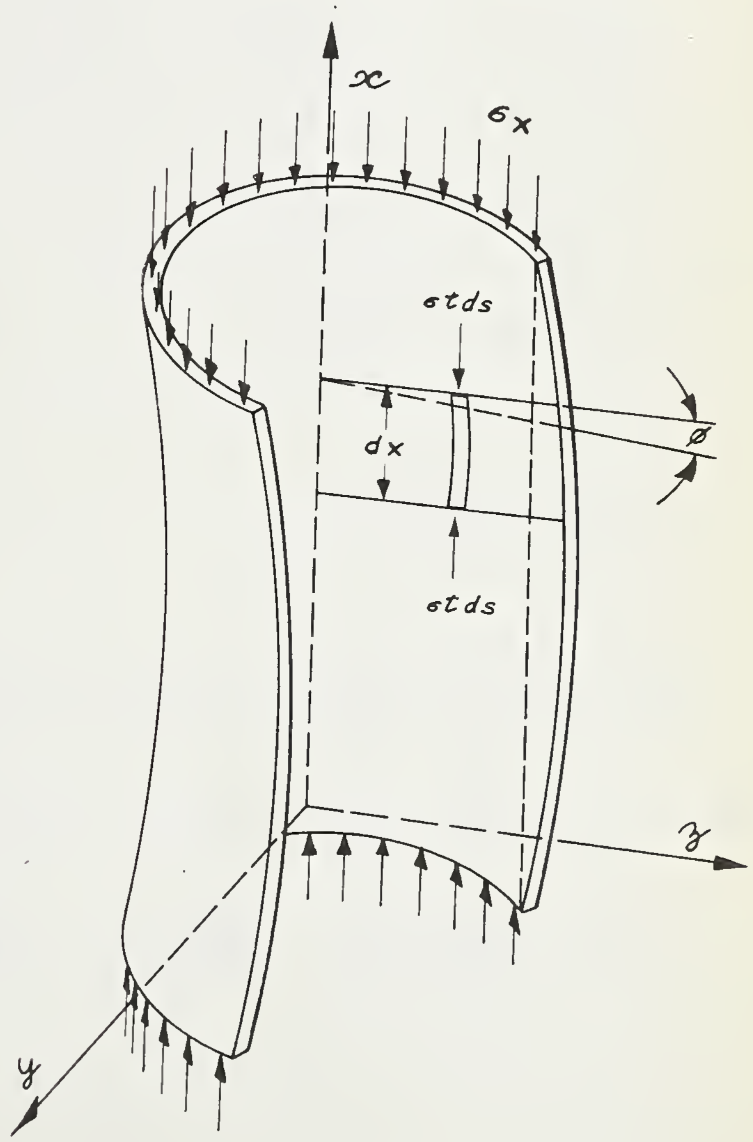


Fig. 2.4 MEMBER OF ARBITRARY UNSYMMETRICAL OPEN CROSS SECTION SUBJECTED TO AXIAL THRUST.

section. The deflection at point y, z is

$$v - \phi z \quad , \quad w + \phi y$$

It follows from the example of the simple strut that the compressive forces acting on the ends of the element dx give the effect of lateral forces

$$-\sigma tds \frac{d^2}{dx^2}\left[v - \phi z\right] \quad , \quad -\sigma tds \frac{d^2}{dx^2}\left[w + \phi y\right]$$

Thus for one strip, the torque per unit length of beam due to the compressive forces is

$$(dw_t)_s = \sigma tds \left[\frac{d^2v}{dx^2} - z\frac{d^2\phi}{dx^2}\right] z - \sigma tds\left[\frac{d^2w}{dx^2} + y\frac{d^2\phi}{dx^2}\right] y$$

Noting that

$$\int_A ytds = c_y A \qquad \int_A ztds = c_z A$$

$$\int_A y^2tds = I_z \qquad \int_A z^2tds = I_y$$

and integrating over the entire cross section gives

$$(w_t)_s = \sigma\frac{d^2v}{dx^2} c_z A - \sigma\frac{d^2w}{dx^2} c_y A - \sigma\frac{d^2\phi}{dx^2} I_y - \sigma\frac{d^2\phi}{dx^2} I_z$$

$$(w_t)_s = Sc_z\frac{d^2v}{dx^2} - Sc_y\frac{d^2w}{dx^2} - \frac{S}{A} I_o \frac{d^2\phi}{dx^2}$$

Now

$$w_t = (w_t)_I + (w_t)_s \, .$$

Thus, it follows that

$$Gc \frac{\partial^2 \phi}{\partial x^2} - Ec_w \frac{\partial^4 \phi}{\partial x^4} = mI_o \frac{\partial^2 \phi}{\partial t^2} - mA \left[c_z \frac{\partial^2 v}{\partial t^2} - c_y \frac{\partial^2 w}{\partial t^2} \right]$$

$$- Sc_z \frac{\partial^2 v}{\partial x^2} + Sc_y \frac{\partial^2 w}{\partial x^2} + \frac{SI_o}{A} \frac{\partial^2 \phi}{\partial x^2}$$

$$Ec_w \frac{\partial^4 \phi}{\partial x^4} + \left[\frac{SI_o}{A} - Gc \right] \frac{\partial^2 \phi}{\partial x^2} + mI_o \frac{\partial^2 \phi}{\partial t^2} - Sc_z \frac{\partial^2 v}{\partial x^2} - mAc_z \frac{\partial^2 v}{\partial t^2}$$

$$+ Sc_y \frac{\partial^2 w}{\partial x^2} + mAc_y \frac{\partial^2 w}{\partial t^2} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 2.8$$

The three simultaneous partial differential equations 2.6, 2.7 and 2.8 define the problem of vibration of a beam of thin-walled open unsymmetrical cross section under axial thrust. These are the equations of triple coupling and are similar to those given by Vlasov (13).

If there is an axis of symmetry, the equations are simplified somewhat. For example, if the z-axis is an axis of symmetry so that $c_y = 0$, then equation 2.7 is independent and represents uncoupled flexural vibrations in the z-direction. Equations 2.6 and 2.8 reduce to the equations for double coupling.

If the beam has two axes of symmetry, then $c_y = c_z = 0$ and all three equations become independent. Equations 2.6 and 2.7 represent uncoupled flexural vibrations in the y-

and z-directions, respectively, and equation 2.8 reduces to the pure torsional vibration equation (14).

2.3 Application of the Method of Separation of Variables

It is possible to reduce the partial differential equations obtained to ordinary differential equations by the method of separation of variables. Put

$$v = V(x)T(t)$$

$$w = W(x)T(t)$$

$$\phi = \Phi(x)T(t)$$

Substituting these expressions into equations 2.6, 2.7 and 2.8 yields the ordinary differential equations

$$\frac{d^2T}{dt^2} + p_n{}^2 T = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 2.9$$

$$EI_\zeta \frac{d^4V}{dx^4} + S\frac{d^2V}{dx^2} - mAp_n{}^2 V - Sc_z \frac{d^2\Phi}{dx^2} + mAp_n{}^2 C_z \Phi$$

$$= 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 2.10$$

$$EI_\eta \frac{d^4W}{dx^4} + S\frac{d^2W}{dx^2} - mAp_n{}^2 W + Sc_y \frac{d^2\Phi}{dx^2} - mAp_n{}^2 c_y \Phi = 0 \quad .. \quad 2.11$$

$$Ec_w \frac{d^4\Phi}{dx^4} + \left(\frac{SI_o}{A} - Gc\right)\frac{d^2\phi}{dx^2} - mI_o p_n{}^2 \Phi - Sc_z \frac{d^2V}{dx^2}$$

$$+ mAc_z p_n{}^2 V + Sc_y \frac{d^2W}{dx^2} - mAp_n{}^2 c_y W = 0 \quad \ldots\ldots \quad 2.12$$

where $p_n{}^2$ is the constant of separation. The equation

2.9 is, of course, the simple harmonic motion differential equation with solution

$$T = A_n \cos p_n t + B_n \sin p_n t$$

It is seen that p_n is the circular frequency of vibration.

2.4 Boundary Conditions

Three types of end support will be considered, with classification according to degree of restraint. A "simple support" for a beam is roughly equivalent physically to support by rollers. It implies restraint against deflection only. For torsional vibrations this means restraint against rotation but not against warping. If there is no restriction against warping, the longitudinal stress σ_x must be zero; then by equation 2.3 $\frac{\partial^2 \phi}{\partial x^2}$ vanishes.* For flexural vibrations a simple support implies restraint against displacement but not against rotation about a transverse axis. The end conditions for a simply supported beam then are

$$v = w = \phi = 0$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

---

* If tests are to be conducted on simply supported members, some way must be found to apply axial loads in such a way that warping is not restrained.

A "fixed support" means that the beam is built-in rigidly. For torsional vibrations this implies restraint not only against rotation but also against warping. If warping is zero, then by equation 2.2, $\frac{\partial \phi}{\partial x}$ must vanish. For flexural vibrations fixed ends imply restraint against rotation about a transverse axis as well as against displacement. The end conditions for fixed supports are

$$v = w = \phi = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} = 0$$

A "free end" implies no restraint of any kind at the end of the beam. For torsional vibrations this means that σ_x must be zero and that the total torque acting on the end of the beam must vanish. The last condition is satisfied by setting $M_t = 0$ in equation 2.4. For flexural vibrations a free end implies that restraining moments and shear forces must be zero at the end. The boundary conditions then are

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\frac{\partial^3 v}{\partial x^3} = \frac{\partial^3 w}{\partial x^3} = 0$$

$$\frac{Gc}{Ec_w}\frac{\partial \phi}{\partial x} - \frac{\partial^3 \phi}{\partial x^3} = 0$$

It is worth noting that if the warping constant c_w is zero, the torsional boundary conditions are somewhat simplified. The function $(\bar{w}_s - w_s)$ is then zero so that the warping displacement u and the longitudinal stress σ_x vanish. The condition of restraint against warping becomes superfluous and a fixed support becomes the same as a simple support, both providing restraint against rotation only. For a free end the only requirement is that the total torque must be zero, since $\sigma_x = 0$ is automatically satisfied. The torsional boundary conditions for a free end then reduce to

$$\frac{\partial \phi}{\partial x} = 0$$

It should also be noted that the boundary conditions for the functions V, W and Φ are the same as those for the deflections v, w and ϕ, respectively, as the boundary conditions must hold for any time t.

2.5 Solution for Simply Supported Ends

If both ends of a beam are simply supported, the boundary conditions are

$$V = W = \Phi = 0 \quad \text{at } x = 0, \ x = 1$$

$$\frac{d^2V}{dx^2} = \frac{d^2W}{dx^2} = \frac{d^2\Phi}{dx^2} = 0 \quad \text{at } x = 0, \ x = 1$$

A set of solutions to the differential equations 2.10, 2.11 and 2.12 which satisfies these boundary conditions is

$$V = v_o \sin \frac{n\pi x}{l}$$

$$W = w_o \sin \frac{n\pi x}{l}$$

$$\Phi = \phi_o \sin \frac{n\pi x}{l}$$

The results obtained upon substituting these expressions back into the differential equations can be simplified if it is observed that the uncoupled natural frequencies of vibration for a simply supported beam are given by

$$p_y^2 = \frac{EI_\zeta}{mA} \left(\frac{n\pi}{l}\right)^4$$

$$p_z^2 = \frac{EI_\eta}{mA} \left(\frac{n\pi}{l}\right)^4$$

$$p_\phi^2 = \left(\frac{n\pi}{l}\right)^2 \left(\frac{n^2\pi^2 Ec_w + l^2 Gc}{l^2 mI_o}\right)$$

It is also convenient to put

$$k_n^2 = p_n^2 + \frac{S}{mA} \left(\frac{n\pi}{l}\right)^2$$

so that

$$p_n = \sqrt{k_n^2 - \frac{S}{mA} \left(\frac{n\pi}{l}\right)^2} \quad \text{..........} \quad 2.13$$

Solution of the differential equations then reduces to solution of the following determinant

$$\begin{vmatrix} p_y^2 - k_n^2 & 0 & c_z k_n^2 \\ 0 & p_z^2 - k_n^2 & -c_y k_n^2 \\ \frac{A}{I_o} c_z k_n^2 & -\frac{A}{I_o} c_y k_n^2 & p_\phi^2 - k_n^2 \end{vmatrix} = 0$$

and thus to solution of the following cubic equation

$$(k_n^2)^3 - \frac{I_o}{I_p} \left[p_y^2 \left[1 - \frac{Ac_y^2}{I_o} \right] + p_z^2 \left[1 - \frac{Ac_z^2}{I_o} \right] + p_\phi^2 \right] (k_n^2)^2$$

$$+ \frac{I_o}{I_p} \left[p_z^2 p_y^2 + p_\phi^2 p_y^2 + p_z^2 p_\phi^2 \right] k_n^2 - p_\phi^2 p_y^2 p_z^2 = 0 \quad \ldots\ldots \quad 2.14$$

Thus for simple support end conditions the natural frequency can be obtained by the solution of the cubic algebraic equation 2.14 for k_n^2 and substitution of the value for k_n^2 into expression 2.13.

2.6 General Solution

The method of solution for any set of boundary conditions will be shown. When it becomes necessary to pick a set of boundary conditions to illustrate the method further,

those corresponding to fixed end supports will be used.

By using the operator $D = \frac{d}{dx}$, equations 2.10, 2.11 and 2.12 can be written in the form

$$\left[\frac{EI_\zeta}{mA} D^4 + \frac{S}{mA} D^2 - p_n^{\,2}\right] V - c_z \left[\frac{S}{mA} D^2 - p_n^{\,2}\right] \Phi = 0 \quad \ldots\ldots \quad 2.15$$

$$\left[\frac{EI_\eta}{mA} D^4 + \frac{S}{mA} D^2 - p_n^{\,2}\right] W + c_y \left[\frac{S}{mA} D^2 - p_n^{\,2}\right] \Phi = 0 \quad \ldots\ldots \quad 2.16$$

$$\left[\frac{Ec_w}{mI_o} D^4 + \left(\frac{S}{mA} - \frac{Gc}{mI_o}\right) D^2 - p_n^{\,2}\right] \Phi - c_z \frac{A}{I_o} \left[\frac{S}{mA} D^2 - p_n^{\,2}\right] V$$
$$+ c_y \frac{A}{I_o} \left[\frac{S}{mA} D^2 - p_n^{\,2}\right] W = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad 2.17$$

A non-trivial solution to this system of ordinary differential equations requires that the following determinant be equal to zero (16).

$$\begin{vmatrix} \frac{EI_\zeta}{mA} D^4 + \frac{S}{mA} D^2 - p_n^{\,2} & 0 & -c_z \left(\frac{S}{mA} D^2 - p_n^{\,2}\right) \\ 0 & \frac{EI_\eta}{mA} D^4 + \frac{S}{mA} D^2 - p_n^{\,2} & +c_y \left(\frac{S}{mA} D^2 - p_n^{\,2}\right) \\ -c_z \frac{A}{I_o} \left(\frac{S}{mA} - p_n^{\,2}\right) & c_y \frac{A}{I_o} \left(\frac{S}{mA} D^2 - p_n^{\,2}\right) & \frac{Ec_w}{mI_o} D^4 + \left(\frac{S}{mA} - \frac{Gc}{mI_o}\right) D^2 - p_n^{\,2} \end{vmatrix} . V = 0$$

Evaluation of the determinant gives

$$\frac{E^3 c_w I_\zeta I_\eta}{m^3 A^2 I_o} D^{12}V + \frac{E^2}{m^3A^2}\left\{ S\left[\frac{c_w I_p}{I_o} + \frac{I_\zeta I_\eta}{A}\right] - \frac{GcI_\zeta I_\eta}{I_o}\right\} D^{10}V$$

$$- p_n^2 \frac{E^2}{m^2A}\left[\frac{c_w I_p}{I_o} + \frac{I_\zeta I_\eta}{A}\right] D^8V$$

$$+ \frac{ES}{m^3A^2}\left\{ S\left[\frac{c_w}{I_o} + \frac{I_p}{A} - \frac{1}{I_o}(c_z^2 I_\eta + c_y^2 I_\zeta)\right] - \frac{GcI_p}{I_o}\right\} D^8V$$

$$- p_n^2 \frac{E}{m^2A}\left\{ 2S\left[\frac{c_w}{I_o} + \frac{I_p}{A} - \frac{1}{I_o}(c_z^2 I_\eta + c_y^2 I_\zeta)\right] - \frac{GcI_p}{I_o}\right\} D^6V$$

$$+ \frac{S^2}{m^3A^2}\left[\frac{SI_p}{AI_o} - \frac{Gc}{I_o}\right] D^6V + p_n^4 \frac{E}{m}\left[\frac{c_w}{I_o} + \frac{I_p}{A} - \frac{1}{I_o}(c_z^2 I_\eta + c_y^2 I_\zeta)\right] D^4V$$

$$- p_n^2 \frac{S}{m^2A}\left[\frac{3SI_p}{I_oA} - \frac{2Gc}{I_o}\right] D^4V$$

$$+ \frac{p_n^4}{m}\left[\frac{3SI_p}{I_oA} - \frac{Gc}{I_o}\right] D^2V - p_n^6 \frac{I_p}{I_o} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 2.18$$

Thus the three fourth order coupled differential equations have been combined to form one twelfth order differential equation. The same equation applies for W and Φ.

The solution to equation 2.18 can be taken as

$$V = \sum_{i=1}^{12} A_i e^{r_i x}$$

$$W = \sum_{i=1}^{12} B_i e^{r_i x}$$

$$\Phi = \sum_{i=1}^{12} C_i e^{r_i x}$$

Substituting the expression for V into equation 2.18 gives the characteristic equation

$$\frac{E^3 c_w I_\zeta I_\eta}{m^3 A^2 I_o}\rho^6 + \frac{E^2}{m^3A^2}\left\{ S\left[\frac{c_w I_p}{I_o} + \frac{I_\zeta I_\eta}{A}\right] - \frac{GcI_\zeta I_\eta}{I_o}\right\}\rho^5$$

$$- p_n^{\,2}\frac{E^2}{m^2 A}\left[\frac{c_w I_p}{I_o} + \frac{I_\zeta I_\eta}{A}\right]\rho^4$$

$$+ \frac{ES}{m^3A^2}\left\{ S\left[\frac{c_w}{I_o} + \frac{I_p}{A} - \frac{1}{I_o}(c_z^{\,2} I_\eta + c_y^{\,2} I_\zeta)\right] - \frac{GcI_p}{I_o}\right\}\rho^4$$

$$- p_n^{\,2}\frac{E}{m^2 A}\left\{ 2S\left[\frac{c_w}{I_o} + \frac{I_p}{A} - \frac{1}{I_o}(c_z^{\,2} I_\eta + c_y^{\,2} I_\zeta)\right] - \frac{GcI_p}{I_o}\right\}\rho^3$$

$$+ \frac{S^2}{m^3A^2}\left[\frac{SI_p}{AI_o} - \frac{Gc}{I_o}\right]\rho^3 + p_n^{\,4}\frac{E}{m}\left[\frac{c_w}{I_o} + \frac{I_p}{A} - \frac{1}{I_o}(c_z^{\,2} I_\eta + c_y^{\,2} I_\zeta)\right]\rho^2$$

$$- p_n^{\,2}\frac{S}{m^2 A}\left[\frac{3SI_p}{I_oA} - \frac{2Gc}{I_o}\right]\rho^2 + \frac{p_n^{\,4}}{m}\left[\frac{3SI_p}{I_oA} - \frac{Gc}{I_o}\right]\rho - p_n^{\,6}\frac{I_p}{I_o} = 0 \qquad \text{................ 2.19}$$

where $\rho = r^2$.

It can be seen from the form of equation 2.19 that the functions V, W, Φ may be written

$$V = A_1 e^{r_1 x} + A_2 e^{-r_1 x} + A_3 e^{r_2 x} + A_4 e^{-r_2 x} + \ldots\ldots + A_{12} e^{-r_6 x} \qquad \text{..2.20}$$

$$W = B_1 e^{r_1 x} + B_2 e^{-r_1 x} + \ldots\ldots\ldots\ldots\ldots\ldots + B_{12} e^{-r_6 x} \qquad \text{......... 2.21}$$

$$\Phi = C_1 e^{r_1 x} + C_2 e^{-r_1 x} + \ldots\ldots\ldots\ldots\ldots\ldots + C_{12} e^{-r_6 x} \qquad \text{......... 2.22}$$

Substituting expressions 2.20 and 2.22 into equation 2.15 gives

$$\left\{\left[\frac{EI_\zeta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2\right] A_1 - c_z \left[\frac{S}{mA} r_1^2 - p_n^2\right] C_1\right\} e^{r_1 x}$$

$$+ \left\{\left[\frac{EI_\zeta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2\right] A_2 - c_z \left[\frac{S}{mA} r_1^2 - p_n^2\right] C_2\right\} e^{-r_1 x}$$

$$+ \left\{\left[\frac{EI_\zeta}{mA} r_2^4 + \frac{S}{mA} r_2^2 - p_n^2\right] A_3 - c_z \left[\frac{S}{mA} r_2^2 - p_n^2\right] C_3\right\} e^{r_2 x}$$

$$+ \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad = \quad 0$$

Substituting expressions 2.21 and 2.22 into equation 2.16 gives

$$\left\{\left[\frac{EI_\eta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2\right] B_1 + c_y \left[\frac{S}{mA} r_1^2 - p_n^2\right] C_1\right\} e^{r_1 x}$$

$$+ \left\{\left[\frac{EI_\eta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2\right] B_2 + c_y \left[\frac{S}{mA} r_1^2 - p_n^2\right] C_2\right\} e^{-r_1 x}$$

$$+ \left\{\left[\frac{EI_\eta}{mA} r_2^4 + \frac{S}{mA} r_1^2 - p_n^2\right] B_3 + c_y \left[\frac{S}{mA} r_2^2 - p_n^2\right] C_3\right\} e^{r_2 x}$$

$$+ \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad = \quad 0$$

As $e^{r_1 x}$, $e^{r_2 x}$, ... are linearly independent,

it follows that

$$\psi_1 = \frac{C_1}{A_1} = \frac{C_2}{A_2} = \frac{\frac{EI_\zeta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2}{c_z \left[\frac{S}{mA} r_1^2 - p_n^2 \right]} \quad \ldots\ldots\ldots\ldots \quad 2.23$$

$$\psi_2 = \frac{C_3}{A_3} = \frac{C_4}{A_4} = \frac{\frac{EI_\zeta}{mA} r_2^4 + \frac{S}{mA} r_2^2 - p_n^2}{c_z \left[\frac{S}{mA} r_2^2 - p_n^2 \right]}$$

$$\chi_1 = \frac{C_1}{B_1} = \frac{C_2}{B_2} = -\frac{\frac{EI_\eta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2}{c_y \left[\frac{S}{mA} r_1^2 - p_n^2 \right]}$$

$$\chi_2 = \frac{C_3}{B_3} = \frac{C_4}{B_4} = -\frac{\frac{EI_\eta}{mA} r_2^4 + \frac{S}{mA} r_2^2 - p_n^2}{c_y \left[\frac{S}{mA} r_2^2 - p_n^2 \right]}$$

Put $\gamma_1 = \frac{\psi_1}{\chi_1}, \quad \gamma_2 = \frac{\psi_2}{\chi_2} \quad \ldots\ldots\ldots\ldots\ldots$

$$\gamma_1 = -\frac{\left[\frac{EI_\zeta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2 \right] c_y}{\left[\frac{EI_\eta}{mA} r_1^4 + \frac{S}{mA} r_1^2 - p_n^2 \right] c_z} \quad \ldots\ldots\ldots\ldots\ldots \quad 2.24$$

Then $C_1 = \psi_1 A_1, \quad B_1 = \gamma_1 A_1, \quad C_2 = \psi_1 A_2 \quad \ldots\ldots\ldots\ldots$

$$V = A_1 e^{r_1 x} + A_2 e^{-r_1 x} + A_3 e^{r_2 x} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$W = \gamma_1 A_1 e^{r_1 x} + \gamma_1 A_2 e^{-r_1 x} + \gamma_2 A_3 e^{r_2 x} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Phi = \psi_1 A_1 e^{r_1 x} + \psi_1 A_2 e^{-r_1 x} + \psi_2 A_3 e^{r_2 x} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Some of the roots ρ of the characteristic equation will be positive and some negative. For negative ρ the function r will be imaginary. The functions V, W and Φ, then, will be combinations of exponential and trigonometric functions. They will be of a form similar to the following:

$$V = A_1 e^{r_1 x} + A_2 e^{-r_1 x} + A_3' \cos r_2 x + A_4' \sin r_2 x + \ldots\ldots$$

$$W = \gamma_1 A_1 e^{r_1 x} + \gamma_1 A_2 e^{-r_1 x} + \gamma_2 A_3' \cos r_2 x + \gamma_2 A_4' \sin r_2 x + \ldots\ldots$$

$$\Phi = \psi_1 A_1 e^{r_1 x} + \psi_1 A_2 e^{-r_1 x} + \ldots\ldots\ldots\ldots$$

where $A_3' = A_3 + A_4$, $A_4' = (A_3 - A_4)\sqrt{-1}$

If the value of the natural frequency p_n were known, it would be possible to solve the characteristic equation 2.19 to ultimately obtain the roots r_i. It would also be possible to evaluate the γ and ψ functions. Expressions for V, W. and Φ of the above form then would be the solutions to the original set of coupled differential equations 2.10, 2.11 and 2.12 to within twelve arbitrary constants A_i which can be obtained from the twelve boundary conditions. For example, if the beam is fixed at both ends, the boundary conditions are

$$V = W = \Phi = 0 \quad \text{at } x = 0,\ x = 1$$

$$\frac{dV}{dx} = \frac{dW}{dx} = \frac{d\Phi}{dx} = 0 \quad \text{at } x = 0,\ x = 1$$

Application of these boundary conditions gives twelve

equations, in all of which the summation of the expressions involving the constants A_i is equated to zero. By determinant theory, a non-trivial solution for any of the constants requires a determinant of the following form to be equal to zero

$$\begin{vmatrix}
1 & 1 & 1 & 0 & \cdots\cdots \\
e^{r_1 l} & e^{-r_1 l} & \cos r_2 l & \sin r_2 l & \cdots\cdots \\
\gamma_1 & \gamma_1 & \gamma_2 & 0 & \cdots\cdots \\
\gamma_1 e^{r_1 l} & \gamma_1 e^{-r_1 l} & \gamma_2 \cos r_2 l & \gamma_2 \sin r_2 l & \cdots\cdots \\
\psi_1 & \psi_1 & \psi_2 & 0 & \cdots\cdots \\
\psi_1 e^{r_1 l} & \psi_1 e^{-r_1 l} & \psi_2 \cos r_2 l & \psi_2 \sin r_2 l & \cdots\cdots \\
r_1 & -r_1 & 0 & r_2 & \cdots\cdots \\
r_1 e^{r_1 l} & -r_1 e^{-r_1 l} & -r_2 \sin r_2 l & r_2 \cos r_2 l & \cdots\cdots \\
\gamma_1 r_1 & -\gamma_1 r_1 & 0 & \gamma_2 r_2 & \cdots\cdots \\
\gamma_1 r_1 e^{r_1 l} & -\gamma_1 r_1 e^{-r_1 l} & -\gamma_2 r_2 \sin r_2 l & \gamma_2 r_2 \cos r_2 l & \cdots\cdots \\
\psi_1 r_1 & -\psi_1 r_1 & 0 & \psi_2 r_2 & \cdots\cdots \\
\psi_1 r_1 e^{r_1 l} & -\psi_1 r_1 e^{-r_1 l} & -\psi_2 r_2 \sin r_2 l & \psi_2 r_2 \cos r_2 l & \cdots\cdots
\end{vmatrix} = 0 \qquad \text{.................} \quad 2.25$$

The above determinant will be known as the boundary condition determinant.

The characteristic equation 2.19 can be solved once the natural frequency p_n is known. The roots of the characteristic equation when substituted into the boundary condition determinant would make its value zero. The differential equations, then, can be solved by a trial and error method which involves solving the characteristic equation and evaluating the boundary condition determinant for various values of the frequency until a frequency is found which makes the value of the determinant zero. This is a natural frequency.

It is quite possible to program a digital computer to solve the differential equations for a particular set of boundary conditions by this trial and error method. This was done for particular cases of unequal and equal leg angle cross sections using the University of Alberta IBM-1620 digital computer programmed in the Fortran II language. The input data for each program was a series of values of thrust and frequency. For each set of values the computer was made to solve the characteristic equation and evaluate the boundary condition determinant. The results from the computer were used to plot the value of the determinant against the frequency for specific values of the thrust to obtain the frequencies corresponding to zero value of the determinant, that is the natural frequencies.

Following are the differential equations and forms of the boundary condition determinant for the particular problems solved.

2.7 Unequal Leg Angle Solution

It was mentioned before that for an angle cross section the warping constant c_w is zero. This reduces the twelfth order differential equation 2.18 to a tenth order differential equation. The characteristic equation is similarly reduced. The functions ψ and γ are the same as before but the 12 x 12 boundary condition determinant is reduced to a 10 x 10 determinant. To be more specific, the last two rows (and the last two columns) of 2.25 are removed.

The differential equation governing vibration of an unequal leg angle is

$$
\begin{aligned}
&\frac{E^2 I_\zeta I_\eta}{m^2 A^2 I_p}\left[\frac{S I_o}{A} - Gc\right] D^{10} V \\
&+ \frac{E}{m^2 A^2}\left\{\frac{S^2}{m}\left[\frac{I_o}{A} - \frac{I}{I_p}\left(I_\eta c_z^2 + I_\zeta c_y^2\right)\right] - \frac{GcS}{m} - p_n^2 \frac{E I_\zeta I_\eta I_o}{I_p}\right\} D^8 V \\
&+ \frac{1}{m^2 A}\left\{\frac{S^2}{mA}\left[\frac{S}{A} - \frac{Gc}{I_p}\right] - p_n^2 E\left(2S\left[\frac{I_o}{A} - \frac{1}{I_p}\left(c_z^2 I_\eta + c_y^2 I_\zeta\right)\right] - Gc\right)\right\} D^6 V \\
&- \frac{p_n^2}{m}\left\{\frac{S}{mA}\left[\frac{3S}{A} - \frac{Gc}{I_p}\right] - p_n^2 E\left[\frac{I_o}{A} - \frac{1}{I_p}\left(c_z^2 I_\eta + c_y^2 I_\zeta\right)\right]\right\} D^4 V \\
&+ \frac{p_n^4}{m}\left[\frac{3S}{A} - \frac{Gc}{I_p}\right] D^2 V - p_n^6 V = 0 \qquad \ldots\ldots\ldots\ldots\ldots \ 2.26
\end{aligned}
$$

2.8 Equal Leg Angle Solution

Because the equal leg angle has an axis of symmetry,

one of the flexural differential equations (2.11) of the three describing the vibration (2.10, 2.11, 2.12) is uncoupled. As with the unsymmetrical angle, the warping constant is zero so that equation 2.12 is only second order in Φ. When the coupled differential equations are combined the result is a sixth order equation. The differential equation for the coupled vibrations is

$$\frac{EI_{\zeta}}{m^2AI_p}\left[\frac{SI_o}{A} - Gc\right] D^6V + \frac{1}{mA}\left[\frac{S^2}{mA} - \frac{GcS}{mI_p} - p_n^{\,2}\frac{EI_{\zeta}I_o}{I_p}\right] D^4V$$

$$- \frac{p_n^{\,2}}{m}\left[\frac{2S}{A} - \frac{Gc}{I_p}\right] D^2V + p_n^{\,4}V = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad 2.27$$

The same equation holds for Φ.

The characteristic equation for the coupled vibrations is

$$\frac{EI_{\zeta}}{m^2AI_p}\left[\frac{SI_o}{A} - Gc\right] \rho^3 + \frac{1}{mA}\left[\frac{S^2}{mA} - \frac{GcS}{mI_p} - p_n^{\,2}\frac{EI_{\zeta}I_o}{I_p}\right] \rho^2$$

$$- \frac{p_n^{\,2}}{m}\left[\frac{2S}{A} - \frac{Gc}{I_p}\right] \rho + p_n^{\,4} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad 2.28$$

The differential equation for the uncoupled vibrations is

$$\frac{EI_{\eta}}{mA} D^4W + \frac{S}{mA} D^2W - p_n^{\,2}W = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2.29$$

Both the equations 2.27 and 2.29 were solved for specific sizes of angle with the aid of the IBM-1620 digital

computer programmed to use the trial and error method outlined. For the coupled vibration equation the function ψ is the same as before (expression 2.23)

$$\psi_1 = \frac{B_1}{A_1} = \frac{B_2}{A_2} = \frac{\frac{EI_\zeta}{mA} r_1^{\,4} + \frac{S}{mA} r_1^{\,2} - p_n^{\,2}}{e\left[\frac{S}{mA} r_1^{\,2} - p_n^{\,2}\right]}$$

If two of the roots of the characteristic equation for the coupled vibrations are real and four are imaginary* the boundary condition determinant will be

$$\begin{vmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ \cos r_1 l & \sin r_1 l & e^{r_2 l} & e^{-r_2 l} & \cos r_3 l & \sin r_3 l \\ \psi_1 & 0 & \psi_2 & \psi_2 & \psi_3 & 0 \\ \psi_1 \cos r_1 l & \psi_1 \sin r_1 l & \psi_2 e^{r_2 l} & \psi_2 e^{-r_2 l} & \psi_3 \cos r_3 l & \psi_3 \sin r_3 l \\ 0 & r_1 & r_2 & -r_2 & 0 & r_3 \\ -r_1 \sin r_1 l & r_1 \cos r_1 l & r_2 e^{r_2 l} & -r_2 e^{-r_2 l} & -r_3 \sin r_3 l & r_3 \cos r_3 l \end{vmatrix} = 0$$

---

* It has not been proven that there will always be two real and four imaginary roots of the characteristic equation. The use of two and four here is as an example only to show the form of the determinant.

It is probably not possible for the roots to be zero. If this happened the form of the determinant would be altered somewhat.

The computer programs were designed to check for real, imaginary and zero roots and set up the boundary condition determinant accordingly.

For the solution of the differential equation for the uncoupled vibrations (2.29) the boundary condition determinant for fixed ends will be as follows if two of the roots are real and two are imaginary

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ \cos r_1 l & \sin r_1 l & e^{r_1 l} & e^{-r_1 l} \\ 0 & r_1 & r_1 & -r_1 \\ -r_1 \sin r_1 l & r_1 \cos r_1 l & r_1 e^{r_1 l} & -r_1 e^{-r_1 l} \end{vmatrix} = 0$$

with a micrometer gave the mean thickness of the second beam as 0.130 inches, although that of the others was 0.125 inches.

3.2 Apparatus

The test apparatus is shown in Figures 3.1 to 3.9. Figures 3.1, 3.2, and 3.3 show the general arrangement while Figures 3.4 to 3.9 give detailed views.

The test apparatus was mounted on a large I-beam (twenty feet long by two feet deep by nine inches wide). This provided a close approximation to a rigid base. The ends of the test beams were bolted between pieces of equal leg angle one-half inch thick by six inches long milled to fit. The lower pieces had a three and one-half inch leg and the upper ones a three inch leg, (see Figures 3.5 and 3.9). Considering the size of the test angle, it would seem that the mathematical boundary conditions for fixed supports should have been closely approximated.

The slide and yoke assembley is shown in Figures 3.4 and 3.5. The base of the clamp was welded to a three-quarter inch thick plate which was milled to fit grooves milled out of one and one-half inch bar. When the grooves were well greased, this gave a slide which moved freely and should have closely satisfied the fixed end conditions. The yoke, slide and clamps were all of mild steel.

Figures 3.4 and 3.5 also show the hydraulic ram, ram

mounting piece and the manual shut-off valve. The ram could be mounted so as to provide either a compressive or tensile force on the beam. The mounting piece was slotted so that the ram position could be adjusted to apply the force through the centroid of the beam cross section. It was necessary to hold the axial load constant while strain gage readings were taken and the vibration frequencies recorded. This was accomplished by closing the manual valve so that there could be no fluid leakage from the ram. Figures 3.4 and 3.5 show a set of mechanical clamps that were intended to be used to maintain the constant load. However, it was found that this device introduced moments to the ends of the beam, so it was abandoned for the hydraulic valve.

As can be seen from Figure 3.3, a pressure gage was incorporated into the hydraulic system to give an indication of the axial force applied to the beam. However, the force was more accurately measured by electrical resistance strain gages bonded to the beam. Baldwin-Lima-Hamilton Corporation (B. L. H.) SR-4 gages were used exclusively. Table 3.2 gives details as to type of gages and bonding agent used for each beam. The strains were read with a B.L.H. Model 120 Strain Indicator. B.L.H. switching and balancing units were also used (see Figure 3.7).

The first and third beams were more than adequately instrumented for the purposes of these tests. Strain gages

were mounted at three stations along the beam; station A about three inches from the slide end, station B at the center and station C about three inches from the fixed end. Table 3.3 gives the exact number of gages located at each station. Figures 3.1, 3.5, 3.8 and 3.9 show details of the strain gage location and orientation.

Tests on the first and third beams revealed that the strains could be accurately measured with a much smaller number of gages. The mean of the readings of the transverse gages at any one station was very little different from the mean at any other station. The gages at station A were usually used during the experiments.

With the axial gages, the mean of the strains given by station A was never significantly different from the mean of that from all the gages. Neither was the mean of the total of eight gages located at the centers of the legs at stations A and C ever far from the mean of all the gages. In both of these cases the deviation was usually less than one per cent of the mean of the total. Further, it was noted in the tests on the third beam that the difference between the reading of the four central gages at station A and the mean of the total was always negligibly small. For the first beam the axial measurements for the experiments were based on the station A readings. For the second and third beams the readings from station A, as well as those

from the four central gages at station C, were used. For the last beam the axial strain reading was based on the mean of the four central gages at station A (see Table 3.2).

For the conversion of the measured strains to stresses to obtain the axial loads, a modulus of elasticity of 29.5×10^6 psi was assumed. Neither the modulus of elasticity nor the modulus of rigidity was obtained experimentally. The effect of variation in these constants was checked theoretically and the results are given in the following chapter.

Two SR-4 strain gages, type A-5-1, were used on each beam to pick up the vibrations. On the symmetrical angles, one pickup was located at the edge of one leg and the other near the center of the beam, while on the unsymmetrical angles one pickup was located at the end of each leg. The electrical signals from the gages were amplified and recorded on a Sanborn Dual Channel Carrier-Amplifier Recorder, Model 321 (see Figures 3.6 and 3.8).

A frequency response check was carried out on the recorder. Known frequencies were fed in from a Hewlett-Packard signal generator and recorded. Table 3.4 shows the results. The deviation in the two lower frequencies is probably not significant; it could come from calibration error in the signal generator or in the setting of the frequency on the generator as well as from the recorder and, at

any rate, is not large. The recorded frequency at 90 cps input, however, is over one per cent low and probably indicates that the inertia of the pens has some effect at that frequency range.

3.3 Test Procedure

Essentially, the hydraulic system was used to put the beam under load, the magnitude of which was measured with the strain gage system. The beam was excited by striking it, and the transient vibrations recorded.

The beams were usually loaded in thousand pound increments from zero to approximately ten thousand pounds in tension, and from zero to near the buckling load in compression.

The process began with a zero load reading of the strain gages. Pressure was then applied to the hydraulic ram, the pressure gage being used to bring the load as close as possible to that desired. The shut-off valve was then closed to hold the load constant and the strain gage readings taken. Next the vibration frequencies were recorded. The beam was struck in several different ways in an attempt to obtain all the lower natural frequencies. It was usually possible to get three, although they were not always fundamentals. After the vibrations were recorded the strain gage readings were again taken. The load was

then removed and a set of zero load strain gage readings taken. The magnitude of the axial load was calculated from the mean of the two sets of readings. This procedure insured that any drift in the zero load reading of the strain gages did not introduce a cumulative error.

TABLE 3.1

TEST SPECIMENS

Beam No.	Nominal Size in inches	Mean Thickness in inches
1	1½ x 1½	0.125
2	2 x 2	0.130
3	2 x 1½	0.125
4	2 x 1½	0.125

TABLE 3.2

STRAIN GAGE DETAILS

Beam No.	No. of Gages	Type	No. Used in Tests	Adhesive
1	46	A-5-1	16	C.I.L. Household Cement
2	20	A-3	20	Eastman 9-10
3	48	A-3	20	Eastman 9-10
4	8	A-5-1-S6	8	Eastman 9-10

TABLE 3.3

STRAIN GAGE LOCATIONS

Beam No.	Station A		Station B		Station C	
	Axial	Transverse	Axial	Transverse	Axial	Transverse
1	12	4	10	4	12	4
2	12	4	0	0	4	0
3	12	4	12	4	12	4
4	4	4	0	0	0	0

TABLE 3.4

FREQUENCY RESPONSE CHECK OF SANBORN RECORDER

Input Frequency cps	Recorded Frequency cps	Deviation
30.0	29.8	0.7 %
60.0	60.2	0.3 %
90.0	88.8	1.3 %

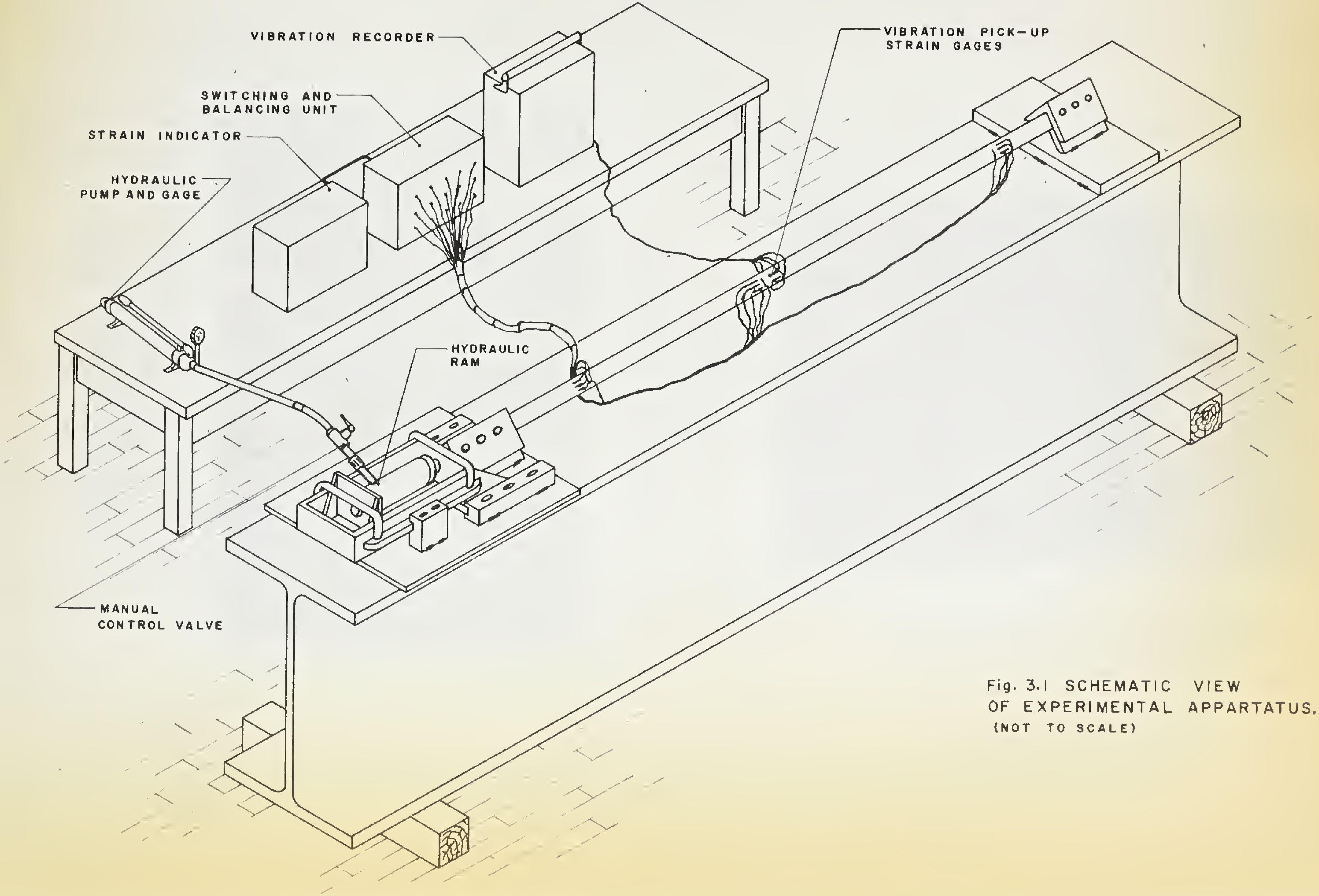


Fig. 3.1 SCHEMATIC VIEW OF EXPERIMENTAL APPARTATUS.
(NOT TO SCALE)

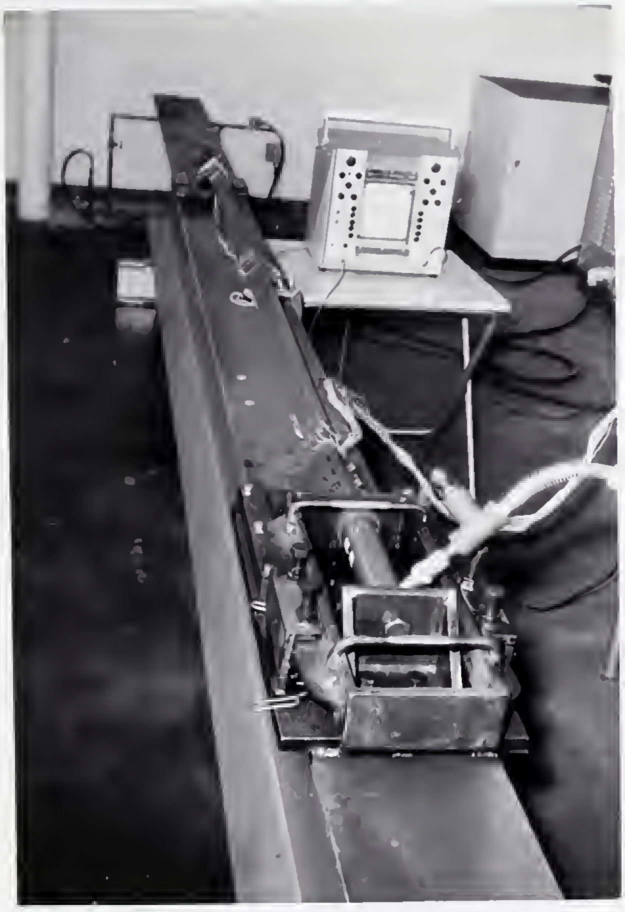


FIGURE 3.2

EXPERIMENTAL APPARATUS

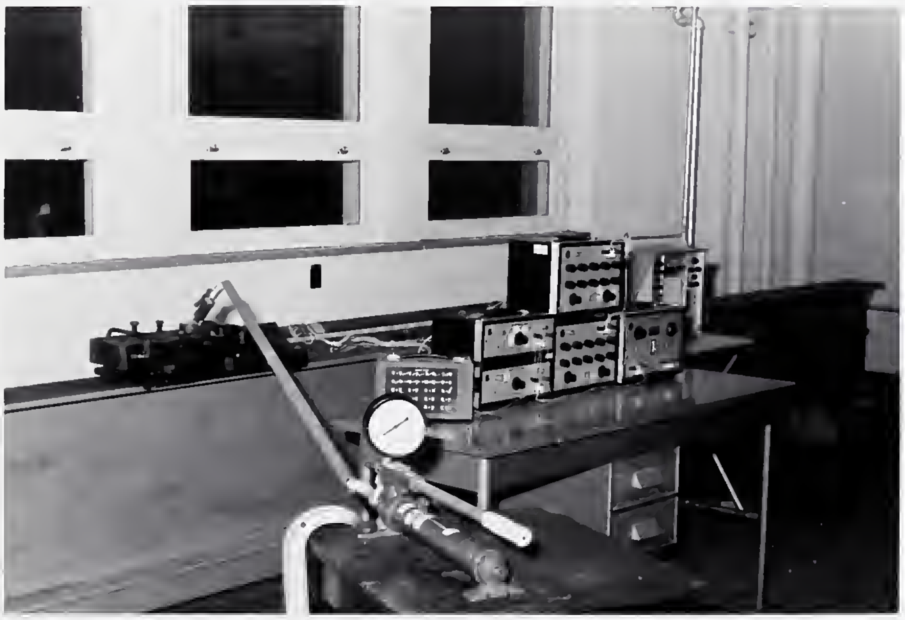


FIGURE 3.3

GENERAL VIEW OF APPARATUS

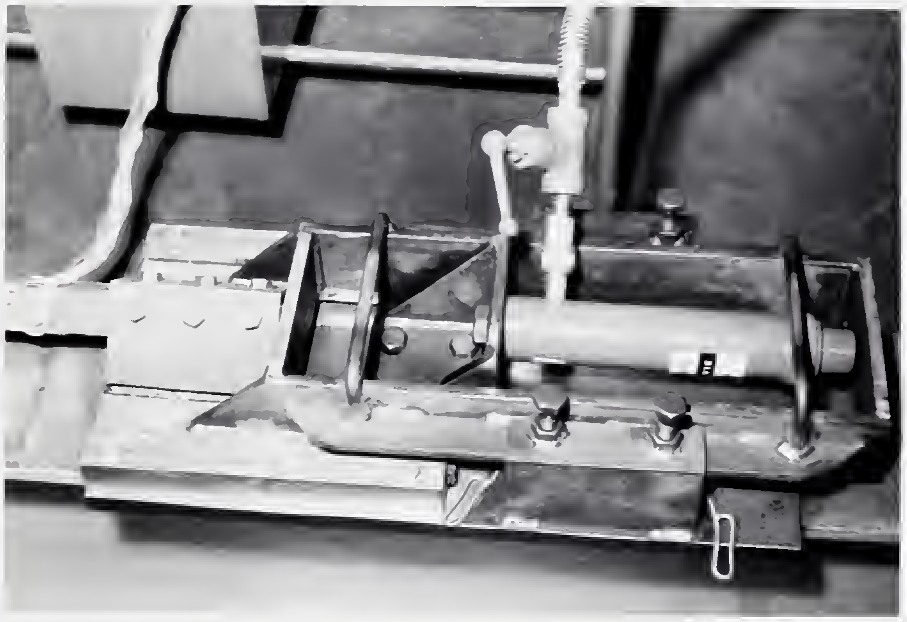


FIGURE 3.4

SLIDE AND YOKE ASSEMBLY, HYDRAULIC RAM AND VALVE

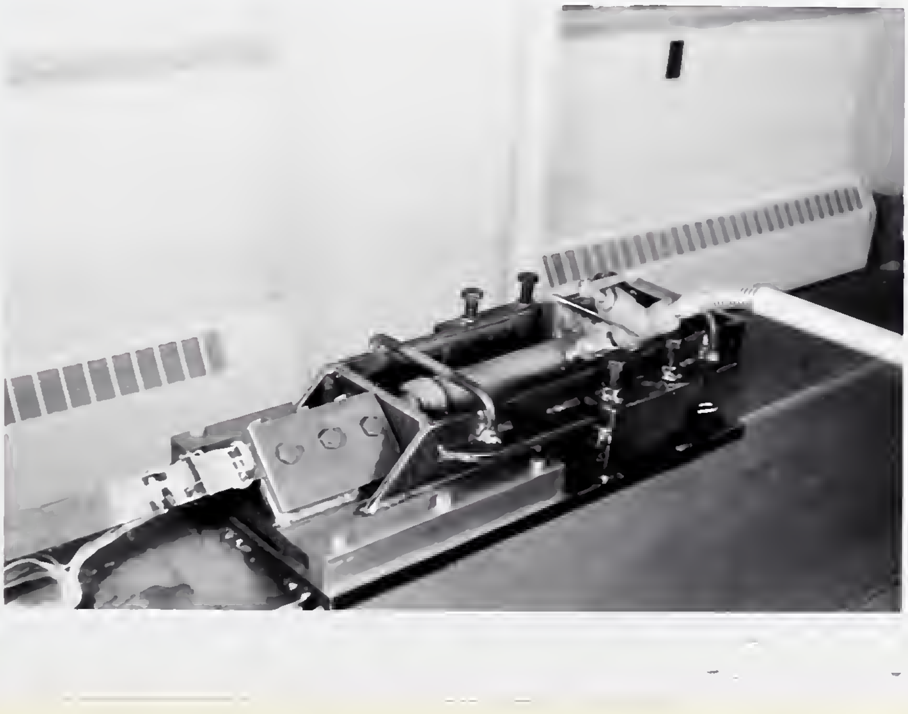


FIGURE 3.5

SLIDE ASSEMBLY

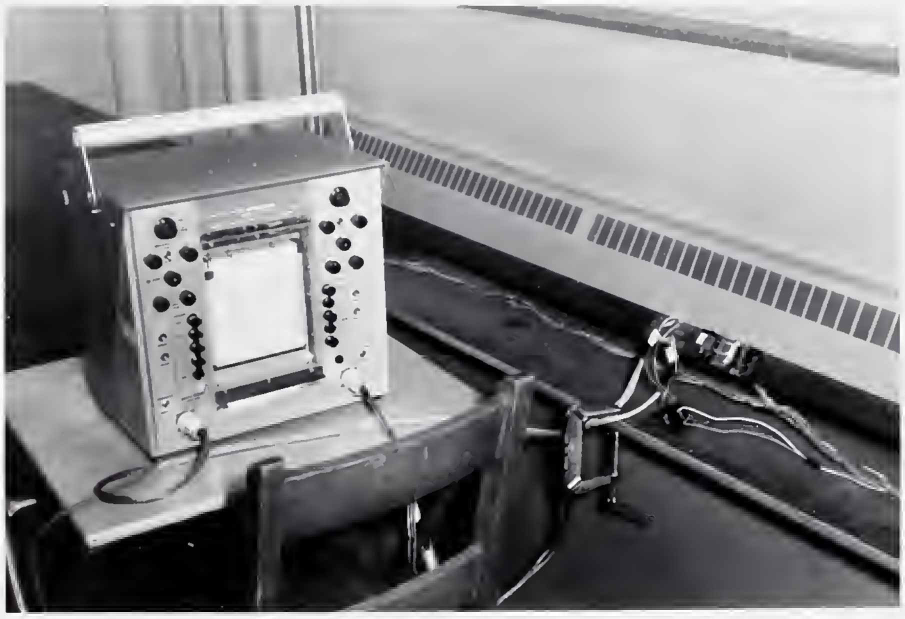


FIGURE 3.6

VIBRATION RECORDING EQUIPMENT

FIGURE 3.7

STRAIN INDICATOR AND SWITCHING UNITS
FOR MEASUREMENT OF AXIAL LOAD

FIGURE 3.8

DETAILS OF STRAIN GAGE ATTACHMENT

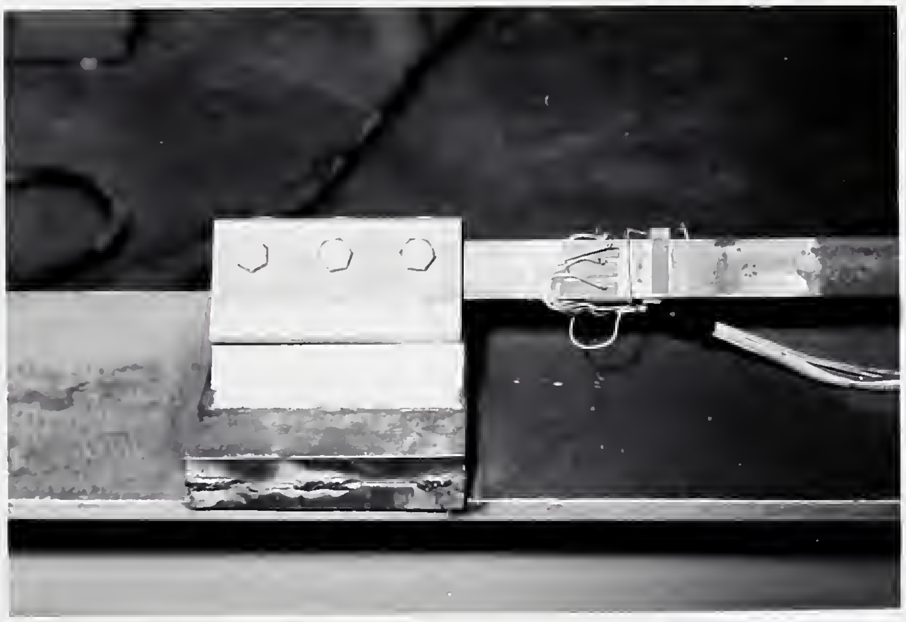


FIGURE 3.9

FIXED END CLAMP

CHAPTER IV

RESULTS AND DISCUSSION

In the following pages, tables and graphs are presented which show the results of theoretical and experimental investigation into the effect of axial load on vibration frequency for beams of both symmetrical and unsymmetrical angle cross section.

4.1 Simply Supported Ends

As is shown in section 2.5, the deflected form of a beam with simply supported ends may be taken as a sine wave function, and the axial thrust-vibration frequency relation obtained directly from an algebraic equation. Thus, although no experimental work was done on simply supported beams, the problem offered a simple and easy check on the rather complex computer solution. The thrust-frequency relations presented in Table 4.1 and Figures 4.1 and 4.2 were determined for a beam of 1½" x 1½" x 1/8" angle cross section (beam #1) using a desk calculator. For several values of thrust the natural frequencies were also obtained using the computer, by changing the appropriate boundary conditions in computer programs A and B (see Appendix). The two methods gave identical results.

It is interesting to note the form of the curves of

Figures 4.1 and 4.2 for later comparison with those for beams with fixed ends. When vibration frequency is plotted against the axial load, the result is a series of parabolic curves. If the square of the frequency is used, a family of straight lines is obtained. These lines on the frequency squared-axial load plot have the same slope for the three fundamental frequencies. The slope of the lines increase as the order of the harmonic increases. As the differential equations for vibration of beams subjected to axial thrust reduce to those for buckling when the frequency is set equal to zero, the curves cross the thrust axis at the buckling load.

4.2 Theoretical Results, Fixed Ends

The results obtained by solution of the differential equations for fixed end boundary conditions for beam #1 are presented in Table 4.2 and in Figures 4.3 and 4.4. Table 4.4 and Figure 4.5 show the effect that variation in the modulus of elasticity E and the modulus of rigidity G have on the relationships between axial load and frequency for beam #1. For beam #2 (2" x 2" x 1/8" angle cross section) the theoretical results are given in Table 4.5 and Figures 4.6 and 4.7. Table 4.7 and Figures 4.8 and 4.9 present the solutions for the unsymmetrical beams (#3 and #4).

These results were obtained using the methods outlined

in sections 2.6, 2.7 and 2.8 and the University of Alberta IBM-1620 digital computer. The computer source programs in the Fortran II language as well as the program input data for the particular beams are included in the appendix.

The curves for the fixed end boundary conditions are similar to those for simply supported ends in that the general form is the same and that they still cross the thrust axis at the buckling loads. However close inspection reveals differences. This can be most clearly seen in the axial load-square of frequency plots. While most of the loci appear to be straight lines as they are for simple supports, the upper coupled fundamental frequency of beam #2 (Figure 4.7) shows definite curvature. It is also noticeable that the curves for the fundamental frequencies no longer have the same slopes. Possibly these differences are the result of the more complicated mathematical relationships introduced by the fixed end boundary conditions. Also, the corresponding frequencies of vibration are higher with fixed ends than with simple supports. This is because of the increased rigidity introduced by the fixed ends.

The effect on the relationship between axial load and frequency of variation in the elastic moduli E and G is small, as may be seen from Table 4.4 and Figure 4.5, but it is not negligible. For example, for beam #1, a variation in the modulus of elasticity from 28.6×10^6 to 30.0×10^6 psi

causes the uncoupled fundamental frequency for zero thrust to vary over a range of 2.7 per cent of the mean frequency. A change from an E of 28.6 x 10^6 psi and a G of 11.0 x 10^6 psi to an E of 30.0 x 10^6 and a G of 11.9 x 10^6 causes the upper coupled fundamental frequency at zero thrust to vary 3.5 per cent. As would be expected, the frequency increases as the elastic moduli are increased.

It is noticeable that the results for the unsymmetrical beam are small in number. This is partially due to computer time considerations. The solution required several evaluations of the boundary condition determinant for every point obtained and because the determinant was large, in this case (10 x 10), it used a considerable amount of computer time. Obtaining points on the curves in two thousand pound increments, instead of the usual one thousand, saved several hours of computing time with negligible deterioration in the accuracy of the curves.

An additional problem encountered was that in certain areas of the frequency-thrust plane the computer program D would not give solutions. The programs were written so that, in most cases, when a division was to be performed, the possibility of a zero divisor was checked. If the divisor was zero, the set of data was rejected. This limitation had little effect on the symmetrical beam solutions but it did restrict the values that could be obtained

theoretically for the unsymmetrical beams to the point where it was only possible to get four values for the upper coupled frequency. A harmonic frequency is known to exist in the frequency-thrust range of Figures 4.8 and 4.9 but it is not included in the graphs because of the difficulty in obtaining points for it. It is quite possible that these values could be determined with some rewriting of program D, probably of the part which evaluates the determinant.

4.3 Experimental Results

The results obtained with the test apparatus described in the third chapter are given in Table 4.3 and in Figures 4.3 and 4.4 for beam #1, in Table 4.6 and Figures 4.6 and 4.7 for beam #2 and in Table 4.8 and Figures 4.8 and 4.9 for beams #3 and #4.

As the graphs show, it was almost always possible to measure the lower and middle fundamental frequencies over the entire range of axial loads. However, it was often difficult to record the higher frequencies. The author was often unable to strike the beam in such a way as to induce it to vibrate in the higher modes, at least not long enough to obtain a recording from which the frequencies could be accurately measured. The frequencies that could be recorded depended largely on the axial

loading. For high thrust loadings the lower fundamental mode was always very dominant and of large amplitude and this resulted in the higher modes being difficult to obtain. On the other hand, for the high tensile force loadings the beams exhibited a tendency to vibrate in the higher modes so that the frequencies of these modes were relatively easy to record even though the amplitude of vibration was small.

For beam #3 no values of frequencies for thrust loadings are given because the beam was inadvertently buckled before these could be obtained. This is the principal reason that beam #4 was instrumented and tested.

In general, the frequencies obtained experimentally were slightly lower than those obtained theoretically. Several possible reasons for this were investigated. As noted in section 3.2, a check of the calibration of the recorder indicated that the vibrations were accurately recorded except perhaps in the ninety cycle per second frequency range. The possibility of damping affecting the frequency measurements was also checked. As the test consisted of recording transient vibrations, it was quite easy to determine the logarithmic decrement. This was found to be very small so that the effect of damping on the frequency was certainly negligible. Further, it can be seen from Figure 4.5 that the possible variation in the elastic moduli E and G had too small an effect to explain the differences.

The most likely reason for the differences between experimental and theoretical values would seem to lie in the closeness to which the fixed end boundary conditions were approximated by the experimental apparatus. If the ends were not restrained completely, the frequencies measured would be lower than those predicted theoretically on the basis of the fixed boundary conditions, which assumed the ends to be built in rigidly. Even considering the size of the clamps used in relation to the size of the test angles, it is likely that the mathematical boundary conditions were not completely satisfied, although they must have been closely approximated. Then, too, it was necessary to build a slide into the test apparatus in order to apply the axial loads, and this required leaving a certain amount of clearance which was quite possibly the greatest source of error to the fixed end approximation.

There is one thing which casts some doubt on the above argument; it is the fact that for beam #2 the experimental values for the lower coupled fundamental frequency (Figure 4.7) are higher than those predicted by the theory. Beam #2 differs from the other beams tested in one special way. For it the coupled frequencies are such that the corresponding uncoupled torsional frequency is lower than the corresponding uncoupled flexural frequency. The coupled modes of vibration are thus dominated by the torsional mode

whereas the coupled vibrations of the other beams are dominated by the flexural mode. It will be remembered from the discussion of boundary conditions (section 2.4) that for beams of angle cross section for which the warping constant is zero, there is no difference in torsional boundary conditions between a simply supported beam and one with fixed ends. It would thus seem likely that for the coupled modes of vibration for beam #2 the closeness to which the fixed end boundary conditions are approximated would have less effect than it would for the other beams. However, this should tend to bring the theoretical and experimental results into better agreement. It is difficult to see why the theoretical results would correspond to lower frequencies than do the experimental.

4.4 Possible Practical Application

Once the mathematical relation between axial load and frequency of vibration is known it should be possible to measure the axial load on struts by measuring the vibration frequency. There is a variety of equipment available for measuring vibration frequencies so that if the relation between the frequency and the load could be obtained quickly and accurately, there would be many practical applications. For instance, it would then be possible to measure the load on a column in a completed building or to check experimentally

a stress analysis of a complex structure.

It has been shown that the required mathematical relationship can be obtained. In any practical application it would seem logical to base calculations on the lower fundamental frequency as this is the one which is most easily measured. Furthermore, there is the least deviation between experimental and theoretical values for this frequency. Using digital computer programs similar to those included in the appendix, this curve can be obtained quickly even for the case of triple coupling, especially as it always seems to be a straight line on an axial load-square of frequency plot.

However, the matter of accuracy is in some doubt. The results in Table 4.9 indicate the type of error that would be introduced if one were to measure the vibration frequency of the beams tested in this set of experiments, and then calculate the axial load on the basis of the theory presented. This Table was calculated from the lower fundamental frequency curves of Figures 4.4, 4.7 and 4.9. The results are rather poor for small axial loads. Except possibly for the case of beam #2, the accuracies are quite good near the buckling loads, and might even be accepted in some cases in the high tensile load range. With further investigation, this method can probably be made accurate enough in all axial load ranges to be of real practical value.

TABLE 4.1

EFFECT OF AXIAL LOAD ON FREQUENCY
1½" x 1½" x 1/8" ANGLE CROSS
SECTION
SIMPLE SUPPORTS
THEORETICAL RESULTS

Thrust	Uncoupled Fundamental	Uncoupled Harmonic	Lower Coupled Fundamental	Upper Coupled Fundamental
in lb.	Frequencies in cycles/second			
1165	0.0	40.1	19.2	77.8
1000	4.4	41.3	19.7	78.3
0	11.7	46.6	22.5	79.0
-1000	15.9	51.3	25.0	79.7
-2000	19.2	55.8	27.2	80.4
-3000	22.0	59.8	29.2	81.3
-4000	24.6	63.5	31.2	81.8
-5000	26.8	67.2	33.0	82.7
-6000	28.9	70.6	34.8	83.3
-7000	30.6	73.8	36.4	84.2
-8000	32.6	76.8	37.9	84.7
-9000	34.4	79.8	39.4	85.4
-10000	36.2	83.2	40.9	86.1
	Square of Frequency in $(\text{radians/second})^2$			
1165	0.0	63,400	14,600	241,000
1000	759	67,500	15,400	241,700
0	5,370	85,900	20,000	246,300
-1000	9,980	104,400	24,600	251,000
-2000	14,600	122,800	29,200	255,600
-3000	19,200	141,300	33,800	260,200
-4000	23,800	159,700	38,400	264,800
-5000	28,400	178,100	43,000	269,400
-6000	33,000	196,600	47,600	274,000
-7000	37,000	215,000	52,300	278,600
-8000	42,200	233,500	56,900	283,200
-9000	46,900	251,900	61,500	287,800
-10000	51,500	270,400	66,100	292,500

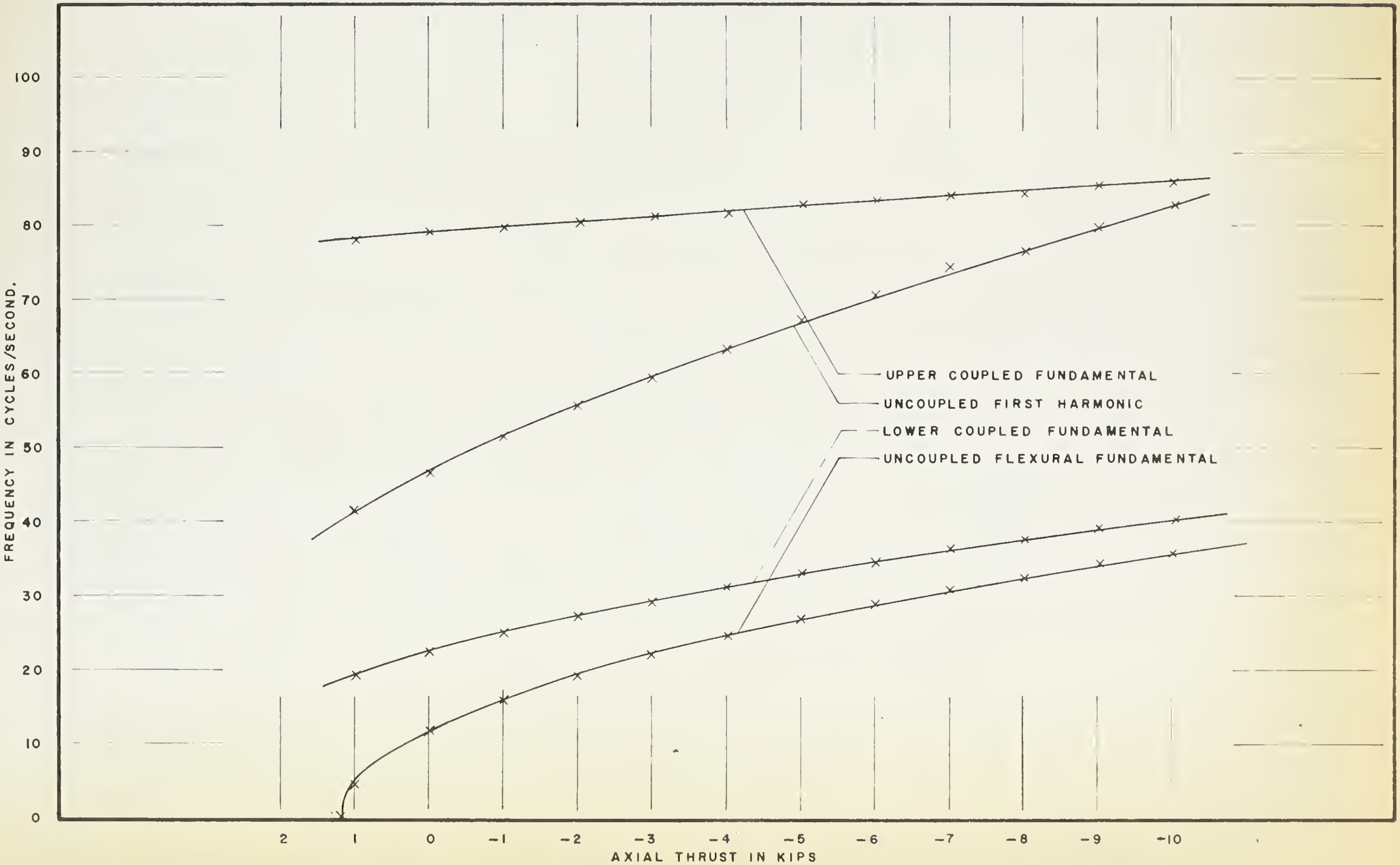


Fig. 4.1 THE EFFECT OF AXIAL LOAD ON VIBRATION FREQUENCY FOR A BEAM OF 1-1/2 x 1-1/2 x 1/8 ANGLE CROSS SECTION WITH SIMPLE SUPPORTS.

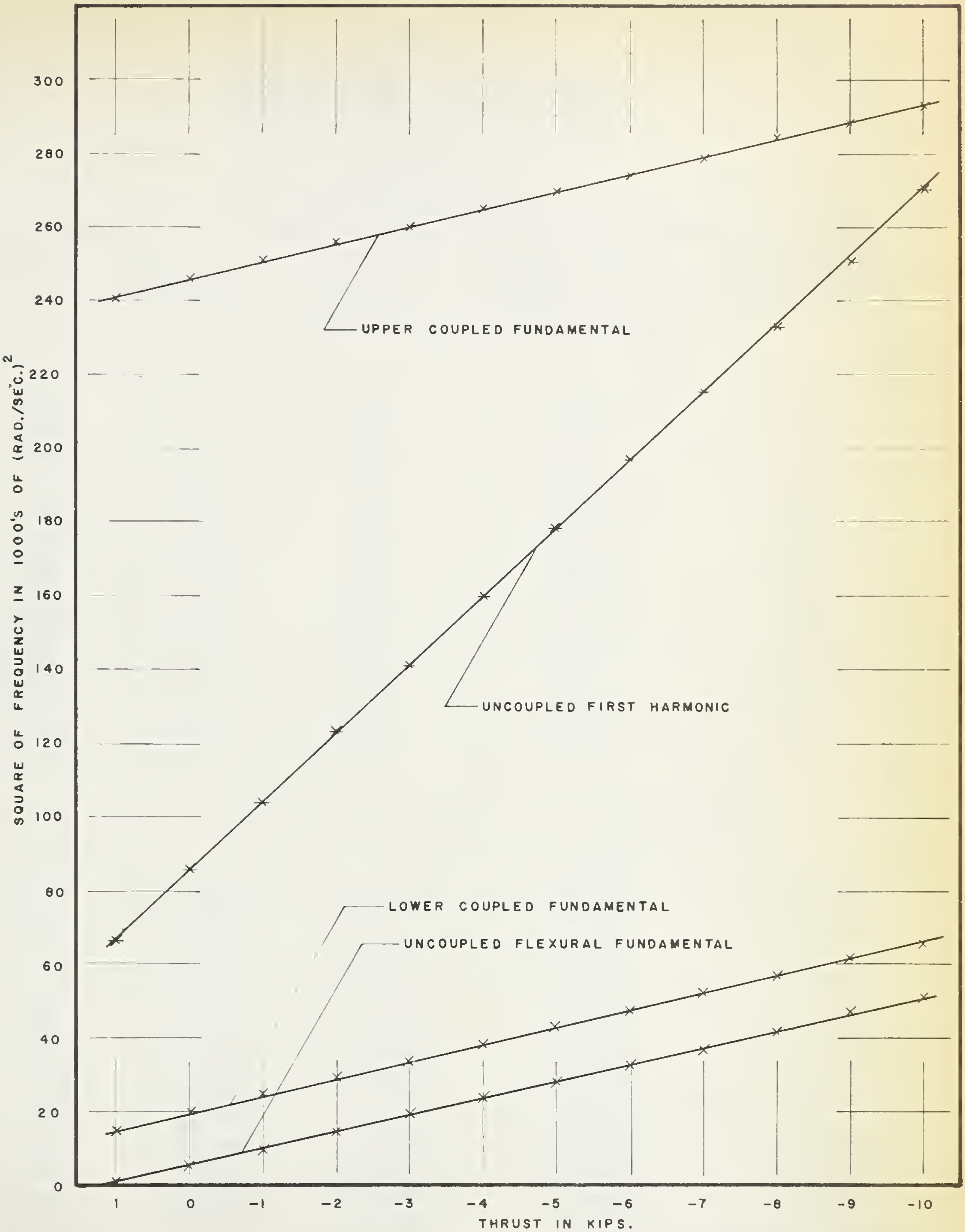


Fig. 4.2. PLOT OF AXIAL LOAD VS SQUARE OF FREQUENCY FOR BEAM OF 1-1/2 x 1-1/2 x 1/8 ANGLE CROSS SECTION WITH SIMPLE SUPPORTS.

TABLE 4.2

EFFECT OF AXIAL LOAD ON FREQUENCY
$1\frac{1}{2}$" x $1\frac{1}{2}$" x 1/8" ANGLE CROSS
SECTION
FIXED ENDS
Beam #1
THEORETICAL RESULTS

Thrust	Uncoupled Fundamental	Uncoupled Harmonic	Lower Coupled Fundamental	Upper Coupled Fundamental
in lb.	Frequencies in cycles/second			
4660	0.0	--	--	--
4000	10.1	55.7	38.2	85.8
3000	16.0	60.6	39.7	87.2
2000	20.2	65.0	41.7	87.9
1000	23.5	69.0	42.8	88.8
0	26.4	72.8	44.4	89.8
-1000	29.0	76.4	45.8	90.8
-2000	31.4	80.0	47.2	91.7
-3000	33.6	83.3	48.6	92.4
-4000	35.9	86.5	49.8	93.2
-5000	37.5	89.5	51.1	94.0
-6000	39.3	92.4	52.4	94.7
-7000	41.1	95.2	53.7	95.7
-8000	42.8	98.0	54.8	96.4
-9000	44.3	100.7	55.8	97.1
-10000	45.8	102.7	56.8	97.8
	Square of Frequency in $(\text{radians/second})^2$			
4000	4,050	123,000	57,600	292,100
3000	10,100	144,800	62,400	299,100
2000	16,000	166,500	68,800	305,500
1000	21,800	188,100	72,800	312,000
0	27,600	209,700	78,000	318,400
-1000	33,300	231,200	83,000	324,600
-2000	39,000	252,600	88,100	330,900
-3000	44,600	274,000	93,200	337,100
-4000	50,200	295,200	98,300	343,100
-5000	55,800	316,300	103,300	349,200
-6000	61,200	337,500	108,300	355,200
-7000	66,700	358,600	113,400	361,200
-8000	72,200	379,700	118,400	367,000
-9000	77,600	400,800	123,400	373,000
-10000	83,000	421,700	128,300	378,800

TABLE 4.3

EFFECT OF AXIAL LOAD ON FREQUENCY
Beam #1
EXPERIMENTAL RESULTS

Thrust	Lower Freq.	Middle Freq.	First Harmonic	Square of L.F.	Square of M.F.	Square of Harmonic
in lb.	cycles/second			(radians/second)^2		
4000	10.5	35.2	--	4,340	48,900	--
3650	12.0	36.8	--	5,680	53,500	--
3000	15.5	38.2	--	9,470	57,600	--
2150	18.7	39.2	62	13,800	60,700	152,000
2000	19.0	39.0	--	14,300	60,100	--
1050	22.0	41.8	--	19,100	69,000	--
1050	22.3	42.0	--	19,600	69,600	--
0	25.2	43.2	70	25,100	73,700	193,000
-1050	27.8	44.0	--	30,500	76,400	--
-2400	30.5	46.0	--	36,700	83,500	--
-3450	32.5	47.5	--	41,700	89,100	--
-4150	33.8	--	83.0	45,100	--	272,000
-4300	34.0	48.3	83.5	45,600	92,100	275,000
-5050	35.5	49.3	85.8	49,700	95,900	291,000
-5150	35.8	49.0	--	50,600	94,800	--
-6050	37.0	50.5	88.0	54,100	101,000	306,000
-6200	38.0	50.5	88.5	57,000	101,000	309,000
-6700	38.5	51.0	90	58,500	103,000	320,000
-7600	39.5	52.5	93	61,600	109,000	341,000
-8650	41	54	96	66,400	115,000	364,000
-9600	43	56	98.5	73,000	124,000	383,000
-10600	45	--	101	80,000	--	403,000

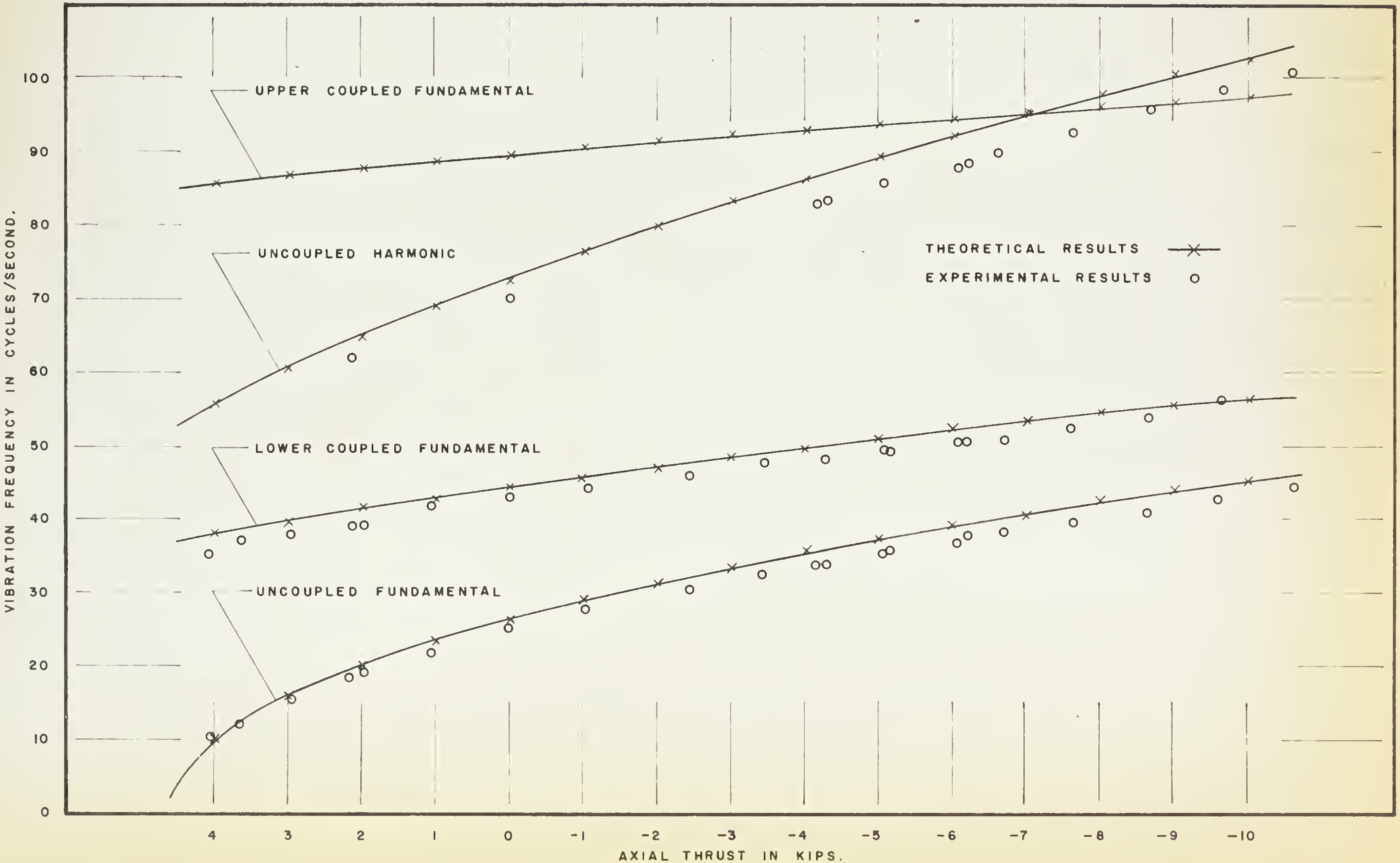
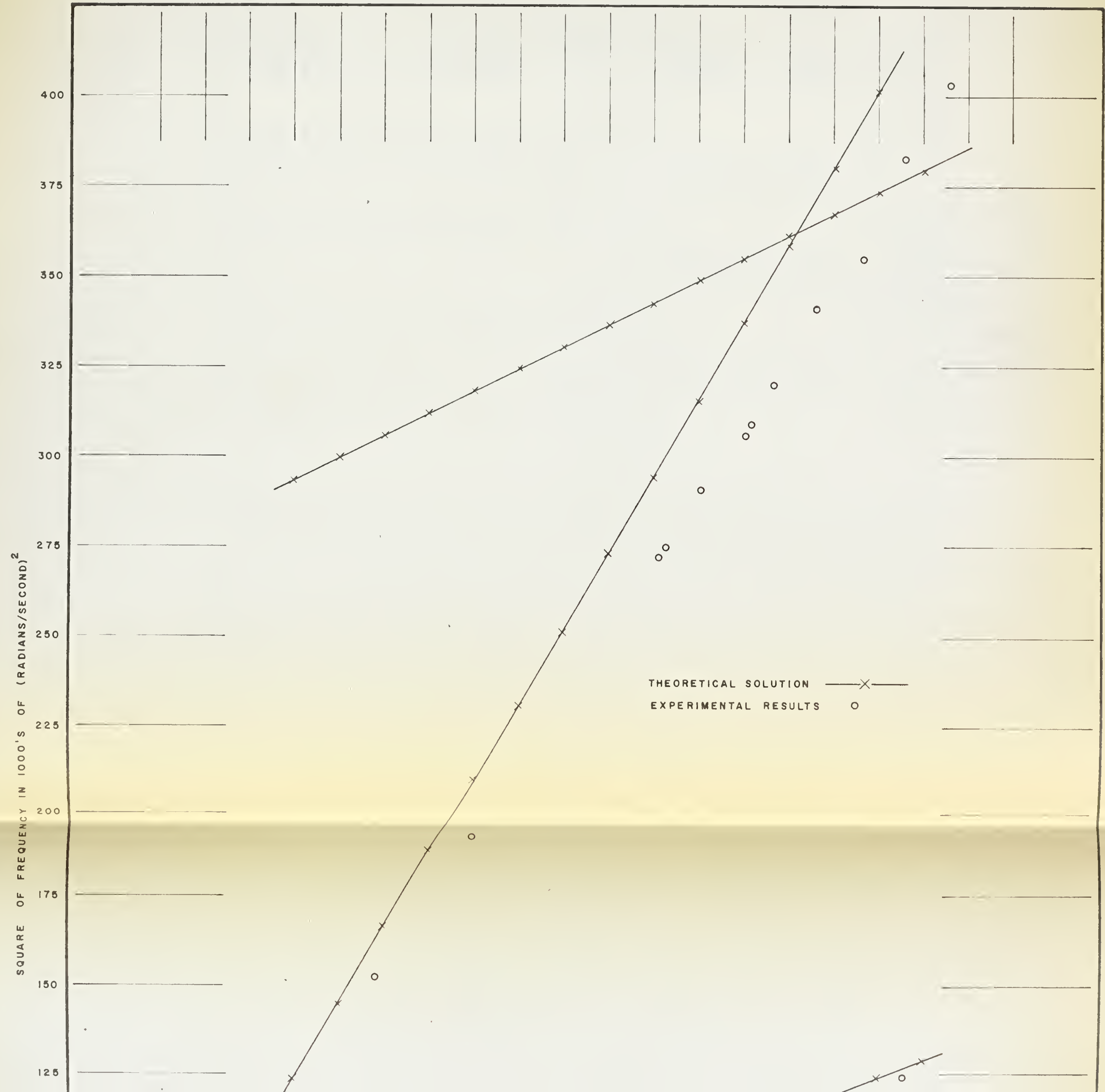


Fig. 4.3 THE EFFECT OF AXIAL LOAD ON VIBRATION FREQUENCY FOR BEAM # 1 (1-1/2 x 1-1/2 x 1/8 ANGLE CROSS SECTION, FIXED ENDS)



400
375
350
325
300
275
250
225
200
175
150
125
SQUARE OF FREQUENCY IN 1000'S OF (RADIANS/SECOND)^2
THEORETICAL SOLUTION
EXPERIMENTAL RESULTS

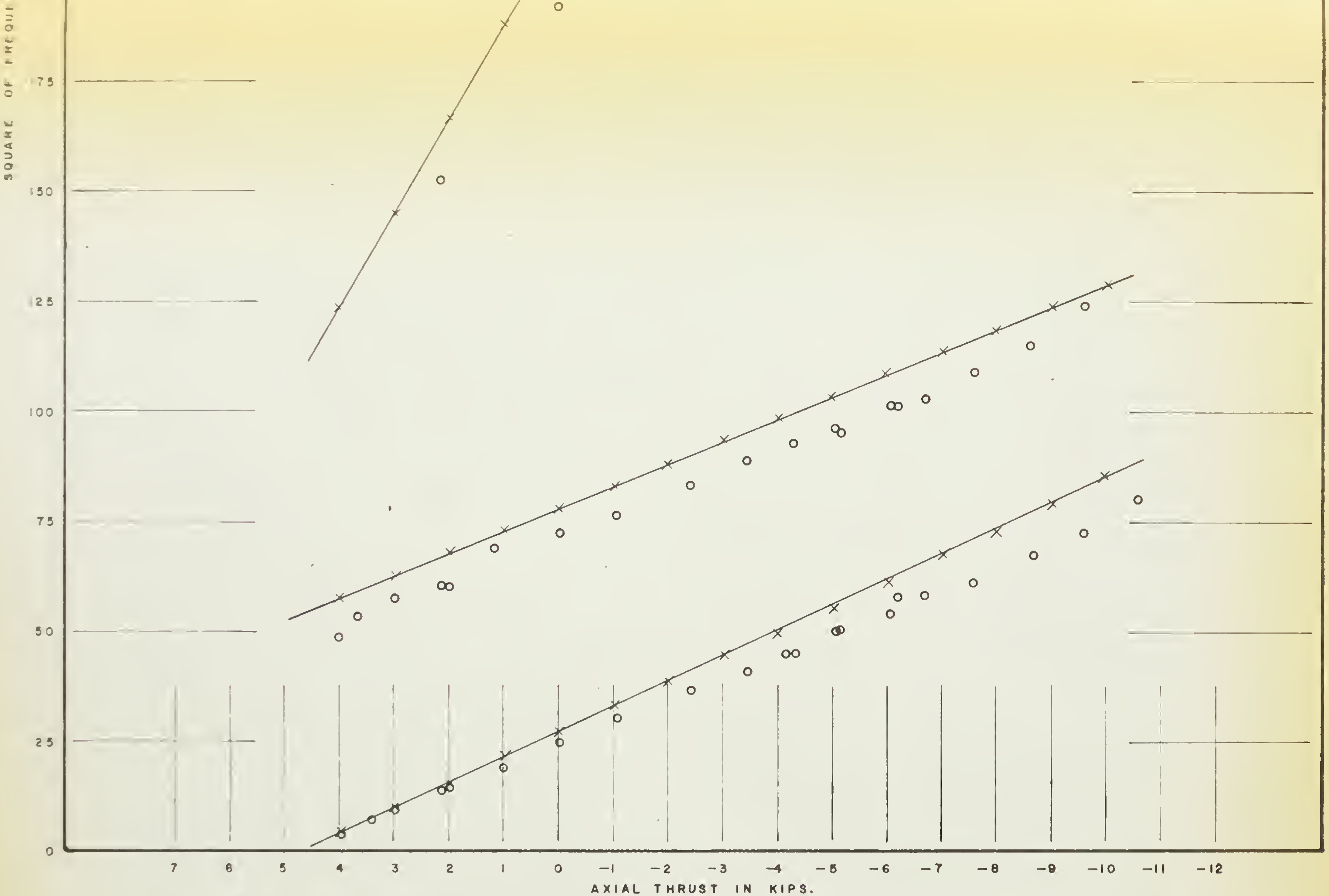


Fig. 4.4 PLOT OF AXIAL LOAD VS. SQUARE OF FREQUENCY FOR BEAM #1 (1-1/2 x 1-1/2 x 1/8 ANGLE CROSS SECTION, FIXED ENDS)

TABLE 4.4

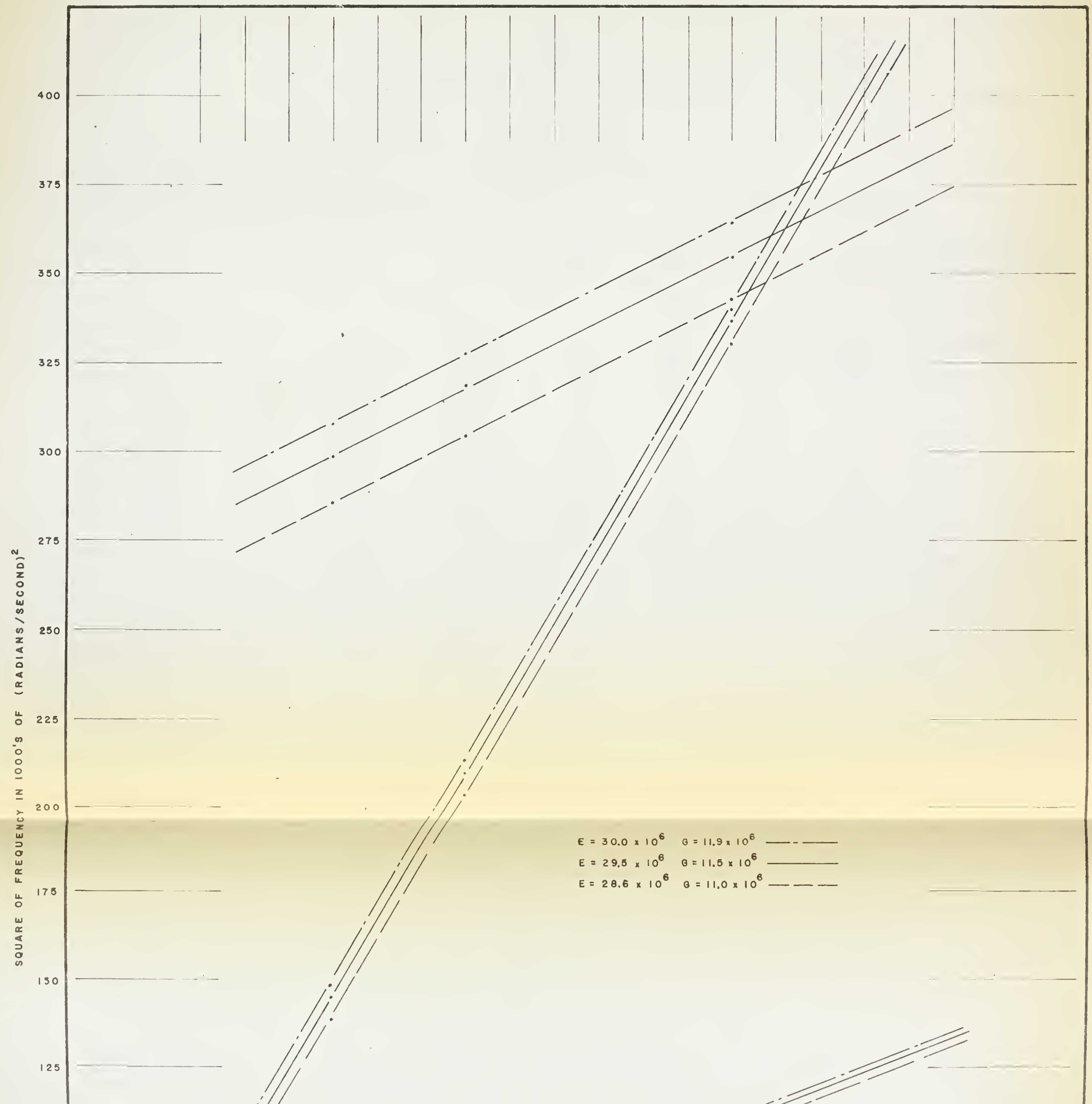
EFFECT OF VARIATION IN ELASTIC CONSTANTS
Beam #1

Uncoupled Frequencies

Thrust	E	Fundamental Frequency	First Harmonic	Square of Fund.	Square of Harm.
in lb.	psi x 10^6	cycles/second		(radians/second)^2	
3000	28.6	15.2	59.2	9,200	138,200
3000	29.5	16.0	60.6	10,100	144,800
3000	30.0	16.4	61.2	10,600	148,200
0	28.6	26.0	71.7	26,800	203,300
0	29.5	26.4	72.8	27,600	209,700
0	30.0	26.7	73.7	28,100	213,200
-6000	28.6	39.0	91.4	60,400	331,000
-6000	29.5	39.4	92.3	61,200	337,000
-6000	30.0	39.5	92.8	61,700	341,000

Coupled Frequencies

Thrust	E	G	Lower Fund.	Upper Fund.	Square of L.F.	Square of U.F.
in lb.	psi	x 10^6	cycles/second		(radians/second)^2	
3000	28.6	11.0	38.8	85.0	59,600	286,000
3000	28.6	11.9	39.4	87.4	61,600	302,400
3000	30.0	11.0	39.6	86.0	62,000	292,000
3000	29.5	11.5	39.7	87.2	62,400	299,100
3000	30.0	11.9	40.3	88.4	64,100	308,000
0	28.6	11.0	43.6	87.8	75,100	305,400
0	28.6	11.9	44.2	90.3	77,300	322,000
0	30.0	11.0	44.2	88.8	77,300	312,000
0	29.5	11.5	44.4	89.8	78,000	318,400
0	30.0	11.9	44.8	90.9	79,600	327,200
-6000	28.6	11.0	51.7	93.2	105,600	343,000
-6000	28.6	11.9	52.2	95.2	107,900	358,400
-6000	30.0	11.0	52.2	93.9	107,600	348,800
-6000	29.5	11.5	52.4	95.0	108,300	355,200
-6000	30.0	11.9	52.8	96.2	110,300	364,000



400
375
350
325
300
275
250
225
200
175
150
125
SQUARE OF FREQUENCY IN 1000'S OF (RADIANS/SECOND)^2
E = 30.0 x 10^6 G = 11.9 x 10^6
E = 29.5 x 10^6 G = 11.5 x 10^6
E = 28.6 x 10^6 G = 11.0 x 10^6

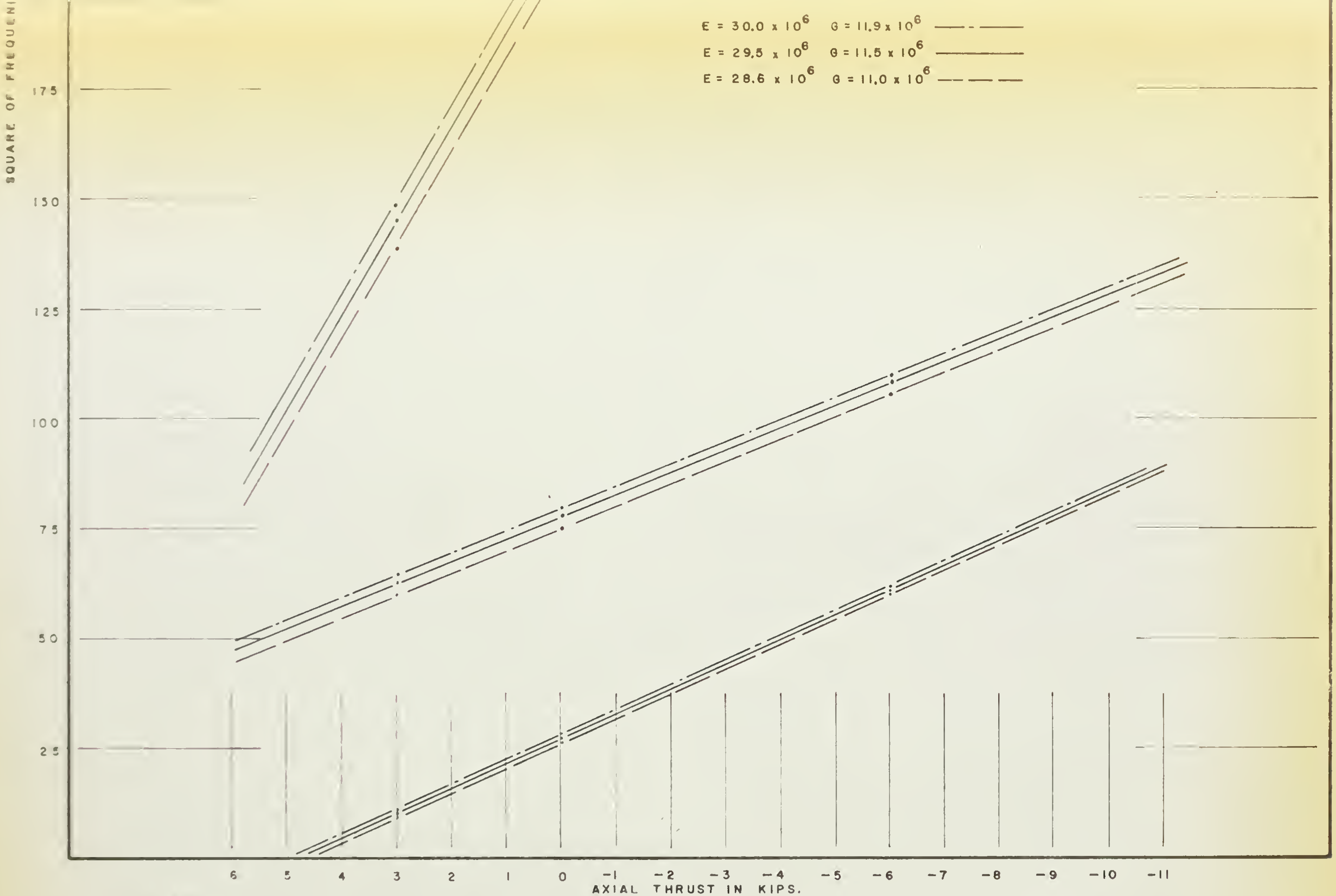


Fig. 4.5 EFFECT OF VARIATION IN E AND G ON AXIAL LOAD VS. SQUARE OF FREQUENCY RELATION FOR BEAM #1.

TABLE 4.5

EFFECT OF AXIAL LOAD ON FREQUENCY
Beam #2
2" x 2" x 1/8" Angle Cross Section
Fixed Ends

THEORETICAL RESULTS

Thrust	Uncoupled Frequencies				Coupled Frequencies					
	Fund. Freq.	First Harmonic	Square of Fund.	Square of Harm.	Lower Fund.	Upper Fund.	First Harmonic	Square of L.F.	Square of U.F.	Square of Harm.
lb.	cps		(rad. /sec.)^2		cps			(rad./sec.)^2		
10000	13.0	73.8	6,700	215,300	31.5	86.2	69.7	39,300	292,800	192,600
9000	16.7	76.3	11,000	231,000	32.9	87.4	72.2	42,700	302,200	206,000
8000	19.7	79.0	15,300	246,600	34.2	88.7	74.6	46,200	311,000	219,200
7000	22.3	81.4	19,600	262,100	35.4	89.8	76.7	49,600	319,000	232,500
6000	24.6	83.8	23,800	277,600	36.6	90.8	78.8	53,000	326,300	245,700
5000	26.6	86.2	28,000	292,600	37.8	91.7	81.0	56,400	333,300	258,900
4000	28.6	88.3	32,200	308,600	38.8	92.8	83.0	59,700	340,000	272,200
3000	30.3	90.5	36,300	324,000	40.0	93.7	84.9	63,100	346,200	285,400
2000	32.0	92.8	40,500	339,300	41.2	94.4	86.8	66,500	352,100	298,600
1000	33.6	94.7	44,600	354,700	41.9	95.3	88.9	69,800	357,900	311,800
0	35.1	97.0	48,700	370,200	43.1	95.7	90.7	73,200	363,400	324,900
-1000	36.5	98.7	52,800	385,400	44.0	96.6	92.4	76,600	368,900	338,200
-2000	37.9	101.2	56,900	400,700	44.8	97.3	94.3	79,900	374,000	351,400
-3000	39.3	102.6	60,900	416,000	45.8	97.8	96.1	83,200	379,200	364,600
-4000	40.6	104.2	65,000	431,300	46.8	98.5	97.8	86,600	384,300	377,900
-5000	41.8	106.2	69,000	446,600	47.7	99.2	99.4	90,000	389,200	391,000
-6000	43.0	108.2	73,000	461,800	48.6	99.8	101.2	93,300	394,000	404,300
-7000	44.2	109.8	77,000	477,000	49.4	100.4	102.8	96,700	398,800	417,500
-8000	45.3	111.7	81,000	492,000	50.3	101.1	104.2	100,000	403,600	430,400
-9000	46.4	113.4	85,000	507,400	51.2	101.7	105.8	103,400	408,200	443,800

TABLE 4.5

EFFECT OF AXIAL LOAD ON FREQUENCY
Beam #2
2" x 2" x 1/8" Angle Cross Section
Fixed Ends

THEORETICAL RESULTS

Thrust	Uncoupled Frequencies				Coupled Frequencies					
	Fund. Freq.	First Harmonic	Square of Fund.	Square of Harm.	Lower Fund.	Upper Fund.	First Harmonic	Square of L.F.	Square of U.F.	Square of Harm.
lb.	cps		$(\text{rad. /sec.})^2$		cps			$(\text{rad./sec.})^2$		
10000	13.0	73.8	6,700	215,300	31.5	86.2	69.7	39,300	292,800	192,600
9000	16.7	76.3	11,000	231,000	32.9	87.4	72.2	42,700	302,200	206,000
8000	19.7	79.0	15,300	246,600	34.2	88.7	74.6	46,200	311,000	219,200
7000	22.3	81.4	19,600	262,100	35.4	89.8	76.7	49,600	319,000	232,500
6000	24.6	83.8	23,800	277,600	36.6	90.8	78.8	53,000	326,300	245,700
5000	26.6	86.2	28,000	292,600	37.8	91.7	81.0	56,400	333,300	258,900
4000	28.6	88.3	32,200	308,600	38.8	92.8	83.0	59,700	340,000	272,200
3000	30.3	90.5	36,300	324,000	40.0	93.7	84.9	63,100	346,200	285,400
2000	32.0	92.8	40,500	339,300	41.2	94.4	86.8	66,500	352,100	298,600
1000	33.6	94.7	44,600	354,700	41.9	95.3	88.9	69,800	357,900	311,800
0	35.1	97.0	48,700	370,200	43.1	95.7	90.7	73,200	363,400	324,900
-1000	36.5	98.7	52,800	385,400	44.0	96.6	92.4	76,600	368,900	338,200
-2000	37.9	101.2	56,900	400,700	44.8	97.3	94.3	79,900	374,000	351,400
-3000	39.3	102.6	60,900	416,000	45.8	97.8	96.1	83,200	379,200	364,600
-4000	40.6	104.2	65,000	431,300	46.8	98.5	97.8	86,600	384,300	377,900
-5000	41.8	106.2	69,000	446,600	47.7	99.2	99.4	90,000	389,200	391,000
-6000	43.0	108.2	73,000	461,800	48.6	99.8	101.2	93,300	394,000	404,300
-7000	44.2	109.8	77,000	477,000	49.4	100.4	102.8	96,700	398,800	417,500
-8000	45.3	111.7	81,000	492,000	50.3	101.1	104.2	100,000	403,600	430,400
-9000	46.4	113.4	85,000	507,400	51.2	101.7	105.8	103,400	408,200	443,800

TABLE 4.6

EFFECT OF AXIAL LOAD ON FREQUENCY
Beam #2

EXPERIMENTAL RESULTS

Thrust	Lower Freq.	Middle Freq.	Upper Freq.	Square of L.F.	Square of M.F.	Square of U.F.
in lb.	cycles/second			(radians/second)^2		
8400	13.3	37.7	--	6,980	56,100	--
7700	16.0	37.9	--	10,100	56,700	--
6650	19.7	39.0	--	15,300	60,100	--
6200	21.6	39.6	--	18,400	61,900	--
5000	23.3	40.2	--	21,300	63,800	--
3900	25.6	41.5	--	25,900	68,000	--
2750	27.2	42.2	--	29,200	70,300	--
2050	28.8	43.0	--	32,800	73,000	--
1200	30.5	43.8	--	36,700	75,700	--
0	32.3	45.1	--	41,200	80,300	--
-1100	33.7	45.8	--	44,800	82,800	--
-1350	34.2	--	91	46,200	--	327,000
-1850	34.7	46.5	--	47,500	85,400	--
-2150	35.4	47.5	91	49,500	89,100	327,000
-2750	36.0	47.8	--	51,200	90,200	--
-3200	36.6	--	92	52,900	--	334,000
-3650	37.5	48.5	--	55,500	92,900	--
-4350	38.2	--	92	57,600	--	334,000
-4900	38.9	49.5	--	59,100	96,700	--
-5100	39.0	--	92.5	60,100	--	338,000
-6050	40.2	50.3	92.5	63,800	99,900	338,000
-6850	41.0	51.1	93	66,600	103,000	342,000
-7750	42	51.5	93.5	69,600	105,000	345,000
-8750	43	52.3	94	73,000	108,000	349,000
-9800	44	52.7	94	76,400	110,000	349,000
-10800	45.3	--	--	81,000	--	--

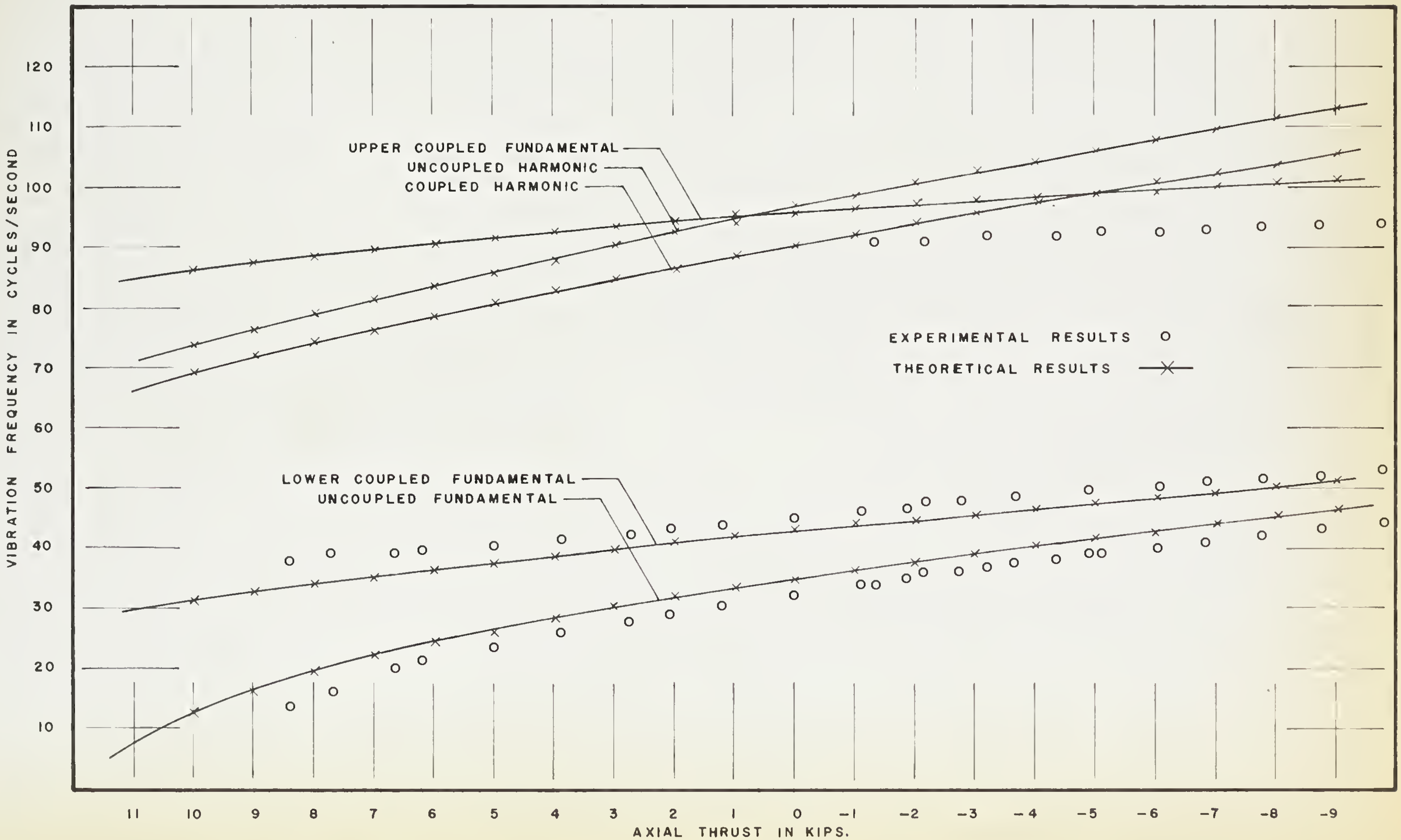
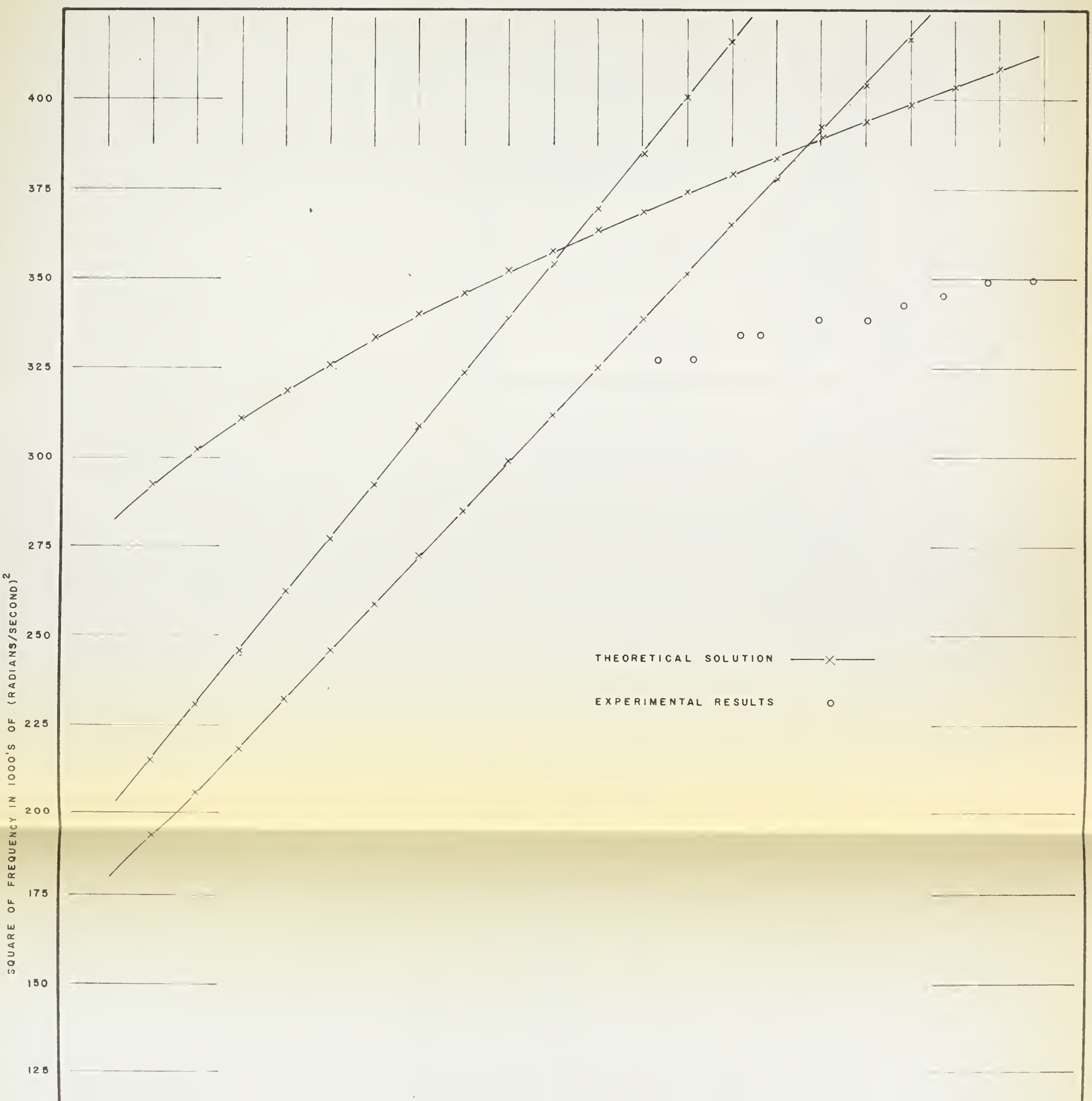


Fig. 4.6 THE EFFECT OF AXIAL LOAD ON VIBRATION FREQUENCY FOR BEAM #2 (2 x 2 x 1/8 ANGLE CROSS SECTION, FIXED ENDS)



SQUARE OF FREQUENCY IN 1000'S OF (RADIANS/SECOND)^2
400
375
350
325
300
275
250
225
200
175
150
125
THEORETICAL SOLUTION
EXPERIMENTAL RESULTS

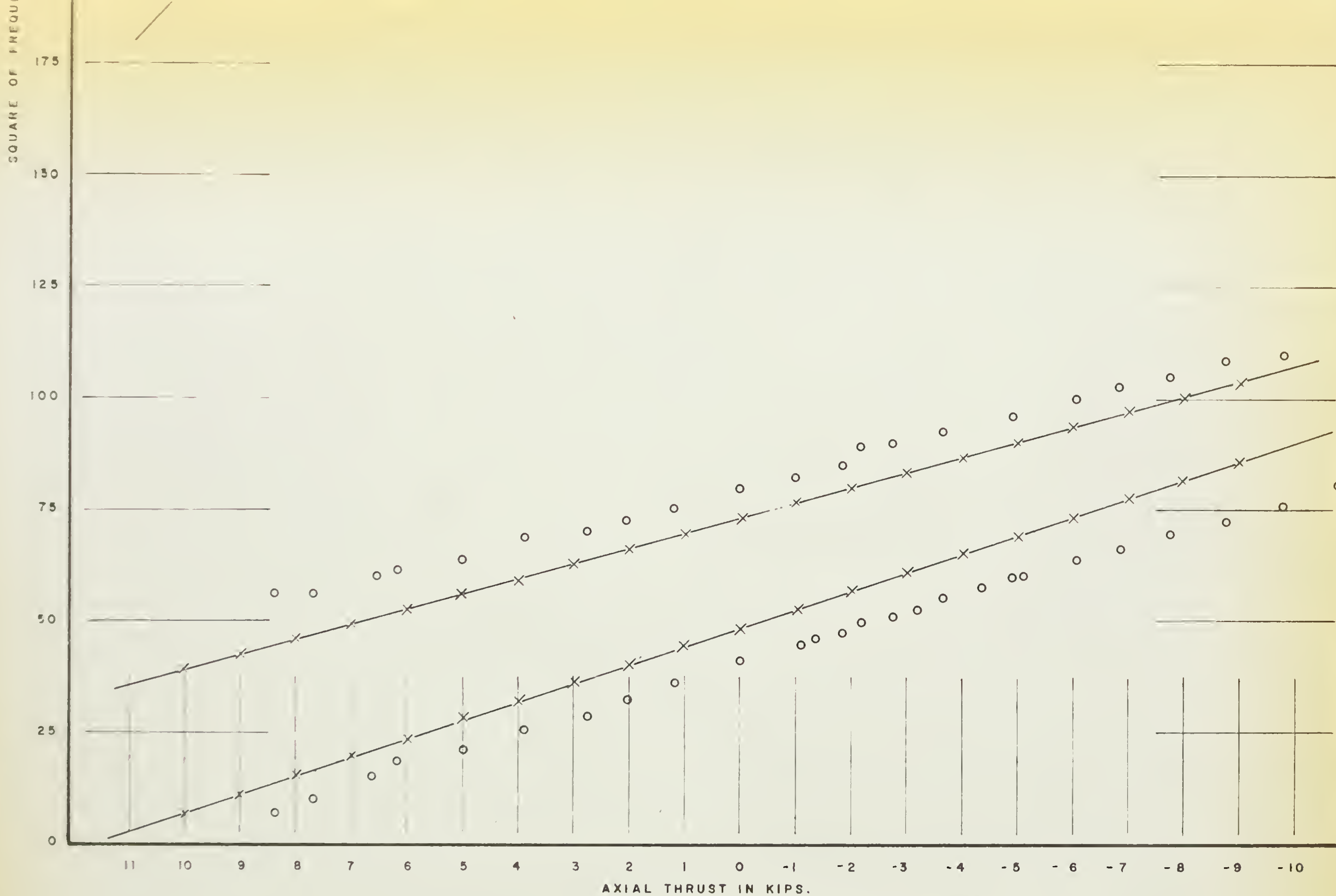


Fig. 4.7 PLOT OF AXIAL LOAD VS. SQUARE OF FREQUENCY FOR BEAM #2 (2 x 2 x 1/8 ANGLE CROSS SECTION, FIXED ENDS.)

TABLE 4.7

EFFECT OF AXIAL LOAD ON FUNDAMENTAL FREQUENCIES

UNSYMMETRICAL BEAMS
THEORETICAL RESULTS

Thrust	Lower Freq.	Middle Freq.	Upper Freq.	Square of L.F.	Square of M.F.	Square of U.F.
in lb.	cycles/second			$(\text{radians/second})^2$		
6000	7.1	39.5	--	2,000	61,400	--
4000	17.4	42.1	84.5	12,000	70,000	283,800
2000	23.6	44.6	86.8	22,000	78,500	297,700
0	28.1	47.0	88.4	31,200	87,200	308,800
-2000	32.3	--	90.3	41,200	--	322,500
-4000	35.8	51.4	--	50,700	104,200	--
-6000	39.0	--	--	60,000	--	--
-8000	45.0	54.8	--	69,200	118,200	--
-10000	49.4	--	--	78,200	--	--

TABLE 4.8

EFFECT OF AXIAL LOAD ON FUNDAMENTAL FREQUENCIES

UNSYMMETRICAL BEAMS

EXPERIMENTAL RESULTS

Thrust	Lower Freq..	Middle Freq.	Upper Freq.	Square of L.F.	Square of M.F.	Square of U.F.
in lb.	cycles/second			(radians/second)^2		
			Beam #3			
0	26.7	46.6	84	28,200	85,700	279,000
-1250	29.0	47.5	85	33,200	89,100	285,000
-2400	31.3	48.8	85.8	38,700	94,000	291,000
-3300	32.7	50.0	87.0	42,200	98,700	299,000
-4250	34.2	50.6	87.5	46,200	101,000	302,000
-5300	35.6	51.5	88.5	50,000	105,000	309,000
-6200	37.0	52.5	89.0	54,000	108,000	313,000
-7400	38.4	53	89.6	58,200	111,000	317,000
-8400	39.8	54	90.0	62,500	115,000	320,000
-9400	41	56	90.8	66,000	124,000	325,000
-10300	42	57	91.5	70,000	128,000	330,000
			Beam #4			
4750	13.4	40.8	82.0	7,100	65,700	265,000
3950	16.7	41.7	82.8	11,000	68,600	271,000
3000	19.7	42.8	83.8	15,300	72,300	277,000
1900	22.7	44.1	84.5	20,300	76,800	282,000
1150	24.6	45.0	85.5	23,900	79,900	289,000
0	26.8	46.2	86.8	28,400	84,300	297,000
-1250	29.7	47.3	87.8	34,800	88,300	304,000
-2100	31.2	48.2	88.5	38,400	91,700	309,000
-3200	32.8	49.4	88.9	42,500	96,300	312,000
-4100	34.4	50.2	89.4	46,700	99,400	316,000
-5250	36.0	51.4	90.0	51,200	104,000	320,000
-6250	37.5	52.5	90.6	55,500	109,000	324,000
-7350	39.0	53.2	91.0	60,000	112,000	327,000
-8250	40.0	53.9	91.4	63,200	115,000	330,000
-9250	41.2	56.0	92.1	67,000	124,000	335,000
-10500	42.8	57.2	93.0	72,300	129,000	341,000

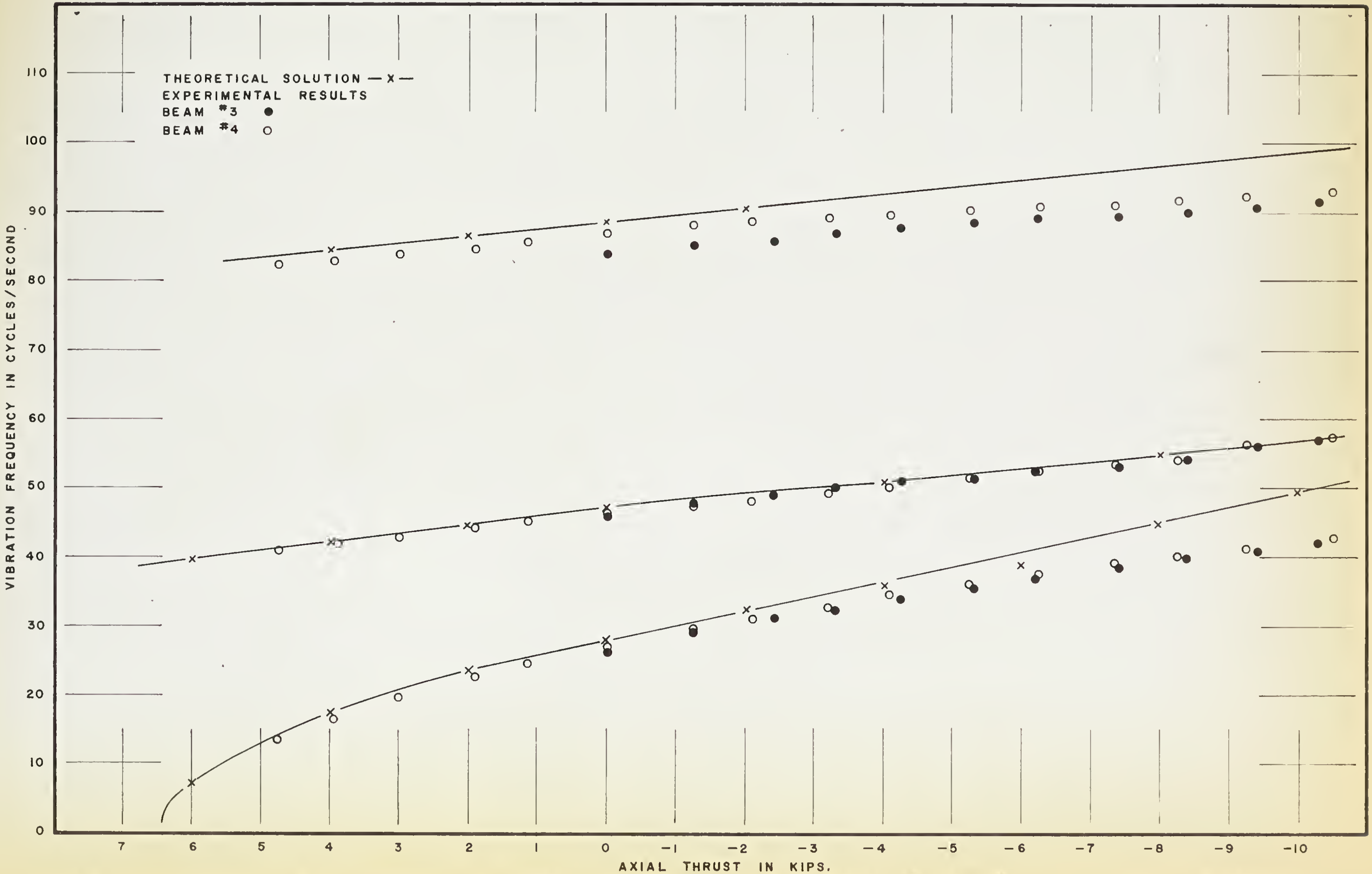
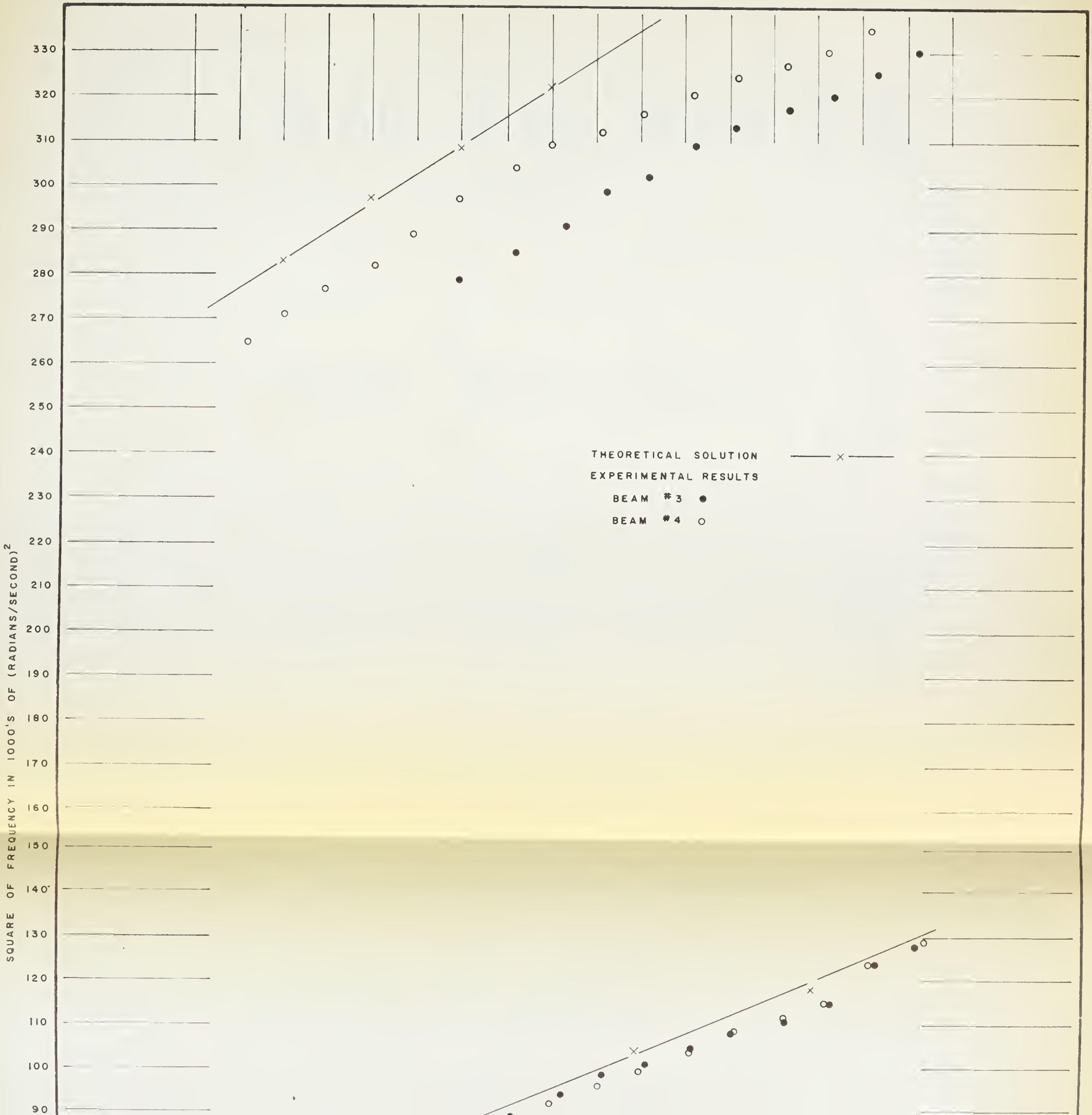


Fig. 4.8 THE EFFECT OF AXIAL THRUST ON VIBRATION FREQUENCY FOR BEAMS #3 AND #4 (2 x 1-1/2 x 1/8 ANGLE CROSS SECTION, FIXED ENDS)



SQUARE OF FREQUENCY IN 1000'S OF (RADIANS/SECOND)^2
330
320
310
300
290
280
270
260
250
240
230
220
210
200
190
180
170
160
150
140
130
120
110
100
90
THEORETICAL SOLUTION
EXPERIMENTAL RESULTS
BEAM #3
BEAM #4

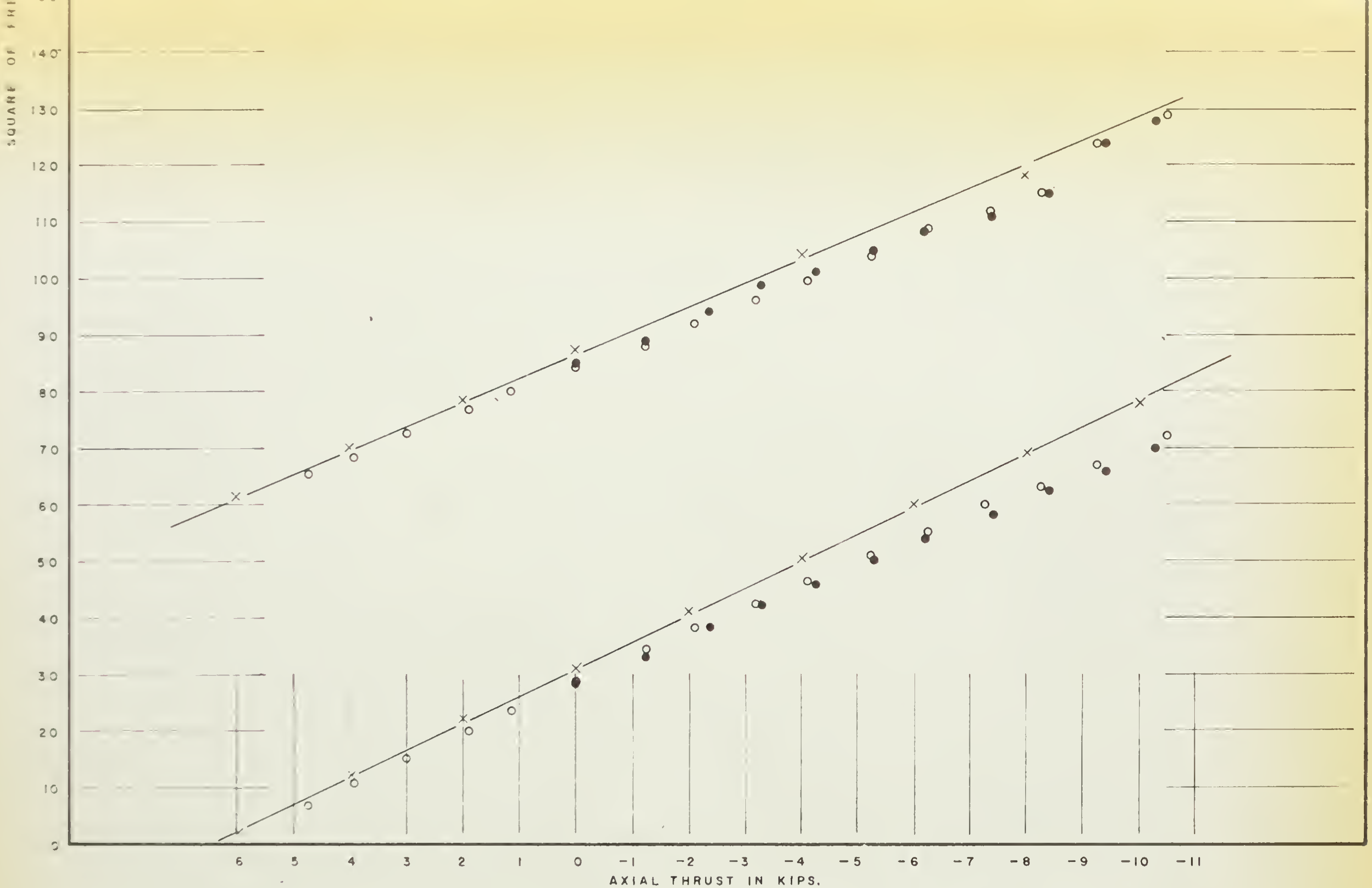


Fig. 4.9 PLOT OF AXIAL LOAD VS. SQUARE OF FREQUENCY FOR BEAMS #3 AND # 4
(2 x 1-1/2 x 1/8 ANGLE CROSS SECTION, FIXED ENDS.)

TABLE 4.9

COMPARISON OF ACTUAL AXIAL LOADS WITH VALUES OBTAINED USING FREQUENCY MEASUREMENTS

Actual Load on Beam	Load on Beam Obtained From Frequency Measurements	Deviation as % of Higher Load (Absolute Value)
lb. Thrust	lb. Thrust	
	Beam #1	
4000	4000	0
3000	3100	3 %
2000	2200	9 %
1000	1400	29 %
0	500	100 %
-5000	-3800	24 %
-10000	-8000	20 %
	Beam #2	
8000	9300	14 %
5000	6600	24 %
2000	4000	50 %
0	500	100 %
-5000	-3800	24 %
-10000	-8000	20 %
	Beams #3, #4	
6000	6000	0
5000	5100	2 %
4000	4200	5 %
2000	2400	17 %
0	600	100 %
-5000	-4000	20 %
-10000	-8500	15 %

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The differential equations governing vibration of thin-walled open section beams subjected to axial load can be solved for other boundary conditions as well as those for simple supports by use of a digital computer.

Application of this method to beams of angle cross section shows that the axial load has an appreciable effect on the natural vibration frequencies. This is verified by the experimental results.

The good correlation between theoretical and experimental results demonstrates the correctness of the method of solution, and verifies the adequacy of the theory with its underlying assumptions, at least for the lower frequencies.

With some care, it is possible to determine the axial loads on structural members by measuring the natural frequency of vibration. Admittedly, the accuracy resulting from such a procedure has been found to be somewhat low in these tests, especially for small axial loads, but this will be improved with more testing and experimental investigation of the end conditions. Certainly more work is required, but this application merits serious consideration.

5.2 Recommendations

Computer program D must be altered so that it is applicable to a more general set of data. It should be possible to rewrite the program so that points on the curves in any region of the axial load-vibration frequency plane can be obtained.

Experiments should be carried out on more beams, of a variety of cross sections. Certainly, another beam of 2" x 2" x 1/8" angle cross section should be tested.

The experimental apparatus should be remodelled so that the slide is clamped tightly during the tests. It would probably be better to obtain the vibration frequencies by use of the Unholtz-Dickie Vibration Testing System now available at the University of Alberta rather than by the methods used for these experiments. The vibration frequencies could probably be obtained more accurately and certainly the higher frequencies would be easier to find using this equipment.

If differences between theoretical and experimental results of the same order as those present now persist after the above-mentioned changes in the experimental apparatus have been made, it should be possible to develop a method of predicting the deviation. On the basis of tests of several beams of a variety of cross sectional dimensions, it can probably be shown to depend, at least

partly, on some non-dimensional parameter such as the thickness to width ratio of the cross section, or the ratio of the uncoupled torsional frequency to the uncoupled flexural frequency.

It might also be possible to investigate the degree to which the fixed end boundary conditions are approximated in physical situations, perhaps by proper placement of strain gages near the ends. It might even be that the deviations can be related to some parameter such as the ratio of amplitude of axial strain at the supports to that at the center of the beam, when it is vibrating.

BIBLIOGRAPHY

1. V. Z. Vlasov, "Thin-walled Elastic Rods", Goz. Izdvo Stroit. Lit., Moscow-Leningrad, 1940.

2. V. Z. Vlasov, "Thin-walled Elastic Beams (Tonkostennye uprugie sterzhni)", State Publishing House for Physico-Mathematical Literature, Moscow, 1959, published in English by the Israel Program for Scientific Translations, Jerusalem, 1961, pp. 386-397.

3. C. F. Garland, "Normal Modes of Vibration of Beams Having Noncollinear Elastic and Mass Axes", Journal of Applied Mechanics, Vol. 7, 1940, pp. 97-105.

4. S. Timoshenko, "Vibration Problems in Engineering", D. Van Nostrand Co. Inc., 1955, pp. 407-411.

5. K. Frederhoffer, "Eigenschwingungen von geralden Staben mit dunnwandigen und offenen Querschnitten", Sitzungsberichte der Weiner Akademie der Wissenschaften, Math-Natur Klasse, Vol. 156 IIa, 1947, pp. 393-416.

6. N. I. Karyakin, "The Torsional Bending Vibrations of Thin-walled Bars", Trudi Belorussk In-ta Inzh Zh.-d., Transp. No. 1, 1957, pp. 147-151.

7. J. M. Gere, "Torsional Vibrations of Beams of Thin-walled Open Section", Journal of Applied Mechanics, Vol. 24, No. 4, Dec. 1954, pp. 381-387.

8. J. M. Gere and J. K. Lin, "Coupled Vibrations of Thin-walled Beams of Open Cross Section", Journal of Applied Mechanics, Vol. 25, No. 3, Sept. 1958, pp. 373-378.

9. Ibid., pp. 373-374.

10. S. Timoshenko, "Theory of Bending, Torsion, and Buckling of Thin-walled Members of Open Cross Section", Journal of the Franklin Institute, Vol. 239, 1945, pp. 201-219, 249-255.

11. S. Timoshenko and J. M. Gere, "Theory of Elastic Stability", McGraw-Hill Co. Inc., 1961, pp. 214-224.

12. Ibid., pp. 225-227, 229-231.

13. V. Z. Vlasov, op. cit., p. 394.

14. J. M. Gere, op. cit., p. 381.

15. S. Timoshenko and J. M. Gere, op. cit., pp. 212-224.

16. L. R. Ford, "Differential Equations", McGraw-Hill Co. Inc., 1955, pp. 141-143.

APPENDIX

FORTRAN SOURCE PROGRAMS

Four different source programs written in the Fortran II language were used, program A for the simple uncoupled flexural vibration problem, program B for the double coupling problem and programs C and D for the triple coupling problem.

The method of solution of any of the three problems is basically the same. The beam constants are read into the computer. Then a value of thrust S and one of the square of the frequency $p_n{}^2$ is read in. The characteristic equation is then set up and the roots obtained. The signs of the squares of the roots are checked. Then, taking proper account of whether the roots are real or imaginary, the boundary condition determinant is set up. Because the chances of finding a zero root are extremely small, an appropriate message is punched out and the data rejected if this ever happens. The boundary condition determinant is evaluated, the values of the thrust, square of frequency and determinant are punched out and the computer proceeds to the next set of values of thrust and frequency.

The computer results are graphed to determine the natural frequencies. The curve obtained by plotting the boundary condition determinant against the frequency, for any given thrust, is continuous. Since, as mentioned in section

2.6, a solution requires that the boundary condition determinant be zero, the natural frequencies correspond to the points where the curve crosses the frequency axis.

The characteristic equation for the problem of uncoupled flexural frequencies is a quadratic algebraic equation. This is solved in program A by the quadratic equation formula.

The characteristic equation for the problem of double coupling is a cubic equation. The first root of this equation is obtained in program B by Newton's Method. The characteristic equation is then reduced to a quadratic and the other roots obtained with the quadratic formula.

The problem of triple coupling was involved enough to make it convenient to write the computer program in two parts. The characteristic equation is set up and solved and the roots punched out in program C. The output from program C is the input to program D. The boundary condition determinant is set up and evaluated in the latter program.

The source programs described used two different subprograms. The solution of the fifth order characteristic equation in program C was written by David Simpson of the University of Alberta Computing Department. The program used to evaluate the determinants was written by Cordell Rolfson of the University of Alberta Computing Department.

Care should be exercised in adapting any of these

source programs to other problems. For example, in program C, the accuracy to which the roots are obtained is governed by the number EPSEL. This must be set to correspond to the range of values of the roots of the characteristic equation. If EPSEL is too large, the roots will not be obtained accurately enough, and if it is too small the roots will not be obtained in a reasonable amount of time.

Following is a list of the notation used in the Fortran source programs. After that comes the source programs themselves. Examples of the input data for the various beams are also given.

Programs A and B

AA	A,	Area of cross section, in square inches
C	c,	Torsion constant, in $(\text{inches})^4$
CI	I_p,	Polar moment of inertia through shear center, in $(\text{inches})^4$
DN	m,	Mass density, in $\frac{\text{lb. sec.}^2}{\text{in.}^4}$
DS	l,	Length of beam, in inches
EC	e,	Eccentricity, in inches
EI	I_η,	Principal centroidal moment of inertia about the η-axis, in $(\text{inches})^4$
EY	E,	Modulus of elasticity, in psi
F	p_n^2	Square of circular frequency, in $(\text{radians/second})^2$

G	G,	Modulus of rigidity, in psi
S	S,	Thrust in pounds
SI	I_o,	Polar moment of inertia about centroid, in $(\text{inches})^4$
ZI	I_ζ,	Principal centroidal moment of inertia about the ζ-axis, in $(\text{inches})^4$

Programs C and D used the same notation as A and B except:

ECY	c_y,	y-coordinate of centroid, in inches
ECZ	c_z,	z-coordinate of centroid, in inches
HA	A,	Area of cross section, in square inches
HC	c,	Torsion constant, in $(\text{inches})^4$
HCI	I_p,	Polar moment of inertia about shear center, in $(\text{inches})^4$

```
..I    PROGRAM A
..I    PROJECT 923121  SYMMETRICAL BEAM
..I    UNCOUPLED FLEXURAL FREQUENCIES
..LOAD FORTRAN EXECUTE
       DIMENSION B(5,5),XM(5),DET(5)
       READ2,EI,AA,DS,EY,DN
  02   FORMAT(1H 3F10.4,2F12.1)
       VA=DN*AA/EY/EI
       VB=0.5/EY/EI
 100   READ16,S,F
  16   FORMAT(1H 2F10.1)
       VC=(S*VB)**2+F*VA
       RTA=S*VB+SQRTF(VC)
       RTB=SQRTF(VC)-S*VB
       RTC=SQRTF(RTA)
       RTD=SQRTF(RTB)
  10   B(1,1)=1.0
       B(1,2)=0.0
       B(1,3)=1.0
       B(1,4)=1.0
       B(2,1)=COSF(RTC*DS)
       B(2,2)=SINF(RTC*DS)
       B(2,3)=EXPF(RTD*DS)
       B(2,4)=1.0/B(2,3)
  14   B(3,1)=0.0
       B(3,2)=RTC
       B(3,3)=RTD
       B(3,4)=-RTD
       B(4,1)=-RTC*B(2,2)
       B(4,2)=RTC*B(2,1)
       B(4,3)=RTD*B(2,3)
       B(4,4)=-RTD*B(2,4)
       N=4
       N2=N-2
       DO90 K=1,N2
       K1=K+1
       K2=K+2
       LARG=K1
       VLARG=B(K,K1)
       VLARG=ABSF(VLARG)
       DO63 J=K2,N
       AJK=B(J,K)
       IF(VLARG-ABSF(AJK))64,63,63
  64   VLARG=ABSF(AJK)
       LARG=J
  63   CONTINUE
       IF(LARG-K1)70,70,67
  67   DO68 I=K,N
       TEMP=B(I,K1)
       B(I,K1)=B(I,LARG)
  68   B(I,LARG)=TEMP
  70   RECIP=1./B(K,K1)
       DO75 J=K2,N
  75   XM(J)=-B(K,J)*RECIP
       DO80 I=K,N
       DO80J=K2,N
```

```
80    B(I,J)=B(I,J)+XM(J)*B(I,K1)
      IF(LARG-K1)85,85,81
81    DO82 J=1,N
      TEMP=B(K1,J)
      B(K1,J)=B(LARG,J)
82    B(LARG,J)=TEMP
85    DO90 J=1,N
      SUM=0.
      DO88 L=K2,N
88    SUM=SUM+XM(L)*B(L,J)
90    B(K1,J)=B(K1,J)-SUM
      DET(1)=B(1,1)
      DO120 M=2,N
      DETM=B(M,1)
      DO  110 I=2,M
110   DETM=B(M,I)*DET(I-1)-DETM*B(I-1,I)
120   DET(M)=DETM
95    PUNCH96,S,F,DET(N)
96    FORMAT(1H 2F10.1,E15.5)
      GO TO 100
      END
```

```
..I    PROGRAM B
..I    PROJECT 923121  SYMMETRICAL BEAM
..I    COUPLED FLEXURAL FREQUENCIES
..LOAD FORTRAN EXECUTE
       DIMENSION RT(3),FN(3),FND(3),B(7,7),XM(7),DET(7)
       READ2,ZI,CI,SI,AA,DS,EC,C,EY,G,DN
  02   FORMAT(1H ,6F10.3/1H ,4F12.1)
       N=6
       VL1=EY*ZI/(DN**2*AA*CI)
       VL2=SI/AA
       VL3=G*C
       VL4=1./(DN*AA)
       VL5=G*C/(DN*CI)
       VL6=EY*ZI*SI/CI
  100  READ16,S,F
  16   FORMAT(1H 2F10.1)
       CF1=VL1*(S*VL2-VL3)
       CF2=VL4*(S*S*VL4-S*VL5-F*VL6)
       CF3=-F*(2.*S*VL4-VL5)
  22   P=CF2/CF1
       Q=CF3/CF1
       R=F*F/CF1
  28   RT1=-R/Q
  30   DO33J=1,150
       FPX=Q+RT1*(2.*P+3.*(RT1))
       IF(FPX)32,31,32
  31   RT1=1.01*RT1
       GOTO33
  32   COR=R+RT1*(Q+RT1*(P+RT1))
       COR=COR/FPX
       RT1=RT1-COR
       IF(ABSF(COR/RT1)-1.0E-05)34,33,33
  33   CONTINUE
  34   BB=P+RT1
       CC=Q+BB*RT1
  35   QR2=(BB*BB-4.*CC)
       QR1=-BB/2.
  38   IF(QR2)41,40,40
  41   QR2=-QR2
       QR=QR2**.5
       RT2=QR1
       RT3=QR
       PUNCH46,S,F,RT1,RT2,RT3
  46   FORMAT(1H 2F10.1,E15.5,4HIMAG,2E15.5)
       GOTO100
  40   QR=QR2**.5
       RT(2)=QR1+QR/2.
       RT(3)=QR1-QR/2.
       VL7=EY*ZI/(DN*AA)
       VL8=S*VL4
       RT(1)=RT1
  50   DO65I=1,3
       FND(I)=VL8*RT(I)-F
  52   FN(I)=(VL7*RT(I)*RT(I)+FND(I))/(EC*FND(I))
       IF(RT(I))54,53,55
  53   DET(N)=666.6
```

```
      GO TO 95
 54   RT(I)=-RT(I)
      RT(I)=RT(I)**.5
      B(1,2*I-1)=1.
      B(1,2*I)=0.
      B(2,2*I-1)=FN(I)
      B(2,2*I)=0.0
      B(3,2*I-1)=COSF(DS*RT(I))
      B(3,2*I)=SINF(DS*RT(I))
      B(4,2*I-1)=FN(I)*B(3,2*I-1)
      B(4,2*I)=FN(I)*B(3,2*I)
      B(5,2*I-1)=0.0
      B(5,2*I)=RT(I)
      B(6,2*I-1)=-RT(I)*B(3,2*I)
      B(6,2*I)=RT(I)*B(3,2*I-1)
      GOTO65
 55   RT(I)=RT(I)**.5
      B(1,2*I-1)=1.0
      B(1,2*I)=1.
      B(2,2*I-1)=FN(I)
      B(2,2*I)=FN(I)
      B(3,2*I-1)=EXPF(DS*RT(I))
      B(3,2*I)=1./B(3,2*I-1)
      B(4,2*I-1)=FN(I)*B(3,2*I-1)
      B(4,2*I)=FN(I)*B(3,2*I)
      B(5,2*I-1)=RT(I)
      B(5,2*I)=-RT(I)
      B(6,2*I-1)=RT(I)*B(3,2*I-1)
      B(6,2*I)=-RT(I)*B(3,2*I)
 65   CONTINUE
      N2=N-2
      DO90 K=1,N2
      K1=K+1
      K2=K+2
      LARG=K1
      VLARG=B(K,K1)
      VLARG=ABSF(VLARG)
      DO63 J=K2,N
      AJK=B(J,K)
      IF(VLARG-ABSF(AJK))64,63,63
 64   VLARG=ABSF(AJK)
      LARG=J
 63   CONTINUE
      IF(LARG-K1)70,70,67
 67   DO68 I=K,N
      TEMP=B(I,K1)
      B(I,K1)=B(I,LARG)
 68   B(I,LARG)=TEMP
 70   IF(B(K,K1))71,100,71
 71   RECIP=1./B(K,K1)
      DO75 J=K2,N
 75   XM(J)=-B(K,J)*RECIP
      DO80 I=K,N
      DO80J=K2,N
 80   B(I,J)=B(I,J)+XM(J)*B(I,K1)
      IF(LARG-K1)85,85,81
```

```
 81   DO82 J=1,N
      TEMP=B(K1,J)
      B(K1,J)=B(LARG,J)
 82   B(LARG,J)=TEMP
 85   DO90 J=1,N
      SUM=0.
      DO88 L=K2,N
 88   SUM=SUM+XM(L)*B(L,J)
 90   B(K1,J)=B(K1,J)-SUM
      DET(1)=B(1,1)
      DO120 M=2,N
      DETM=B(M,1)
      DO  110 I=2,M
110   DETM=B(M,I)*DET(I-1)-DETM*B(I-1,I)
120   DET(M)=DETM
 95   PUNCH96,S,F,DET(N)
 96   FORMAT(1H 2F10.1,E15.5)
      GO TO 100
      END
```

```
..I    PROGRAM C
..I    PROJECT 923124  UNSYMMETRICAL BEAM
..I    SOLUTION OF CHARACTERISTIC EQUATION
..LOAD FORTRAN EXECUTE
       DIMENSION A(50),B(50),C(50)
       READ2,HA,EI,ZI,HCI,SI,ECZ,ECY,HC,EY,G,DN,DS
  02   FORMAT(1H 7F10.5/1H 4F12.2,F10.3)
       PUNCH4,HA,EI,ZI,HCI,SI,ECZ,ECY,HC,EY,G,DN,DS
  04   FORMAT(1H 7F10.5/1H 4E15.5,F10.3)
       VA=(EY*EY*ZI*EI)/(DN*DN*DN*HA*HA*HCI)
       VB=G*HC
       VC=EY/(DN*DN*HA*HA)
       VD=SI/HA-(EI*ECZ*ECZ+ZI*ECY*ECY)/HCI
       VE=EY*ZI*EI*SI/HCI
       VG=1./(DN*DN*HA)
 100   READ28,S,F
       PUNCH28,S,F
  28   FORMAT(1H 2F10.1)
       CFA=VA*(S*SI/HA-VB)
       VI=S*S/DN*VD-VB*S/DN
       VJ=S*S*(S/HA-VB/HCI)/(DN*HA)
       VK=EY*(2.*S*VD-VB)
       VL=S/DN/HA*(3.*S/HA-2.*VB/HCI)
       VM=(3.*S/HA-VB/HCI)/DN
       A(1)=1.0
       A(2)=VC*(VI-F*VE)/CFA
       A(3)=VG*(VJ-F*VK)/CFA
       A(4)=-F/DN*(VL-F*EY*VD)/CFA
       A(5)=F*F*VM/CFA
       A(6)=-F*F*F/CFA
       N=5
       EPSEL=+.100E-10
       N=N+1
       R=0.5
 5     P=0.0
       G=0.0
 6     DO7I=3,N,1
       B(1)=A(1)
       B(2)=A(2)-P*B(1)
 7     B(I)=A(I)-P*B(I-1)-G*B(I-2)
       L=N-2
       DO8 I=3,L,1
       C(1)=B(1)
       C(2)=B(2)-P*C(1)
 8     C(I)=B(I)-P*C(I-1)-G*C(I-2)
       C(N-1)=0.0-P*C(N-2)-G*C(N-3)
       D=C(N-2)**2-C(N-1)*C(N-3)
       IF(D)80,19,80
 80    DELP=(B(N-1)*C(N-2)-B(N)*C(N-3))/D
       DELG=(B(N-1)*C(N-1)-B(N)*C(N-2))/D
       P=P+DELP
       G=G-DELG
       IF(ABSF(DELG)-EPSEL)9,9,6
  9    AB=P**2-4.0*G
       IF(AB)15,10,10
  10   BB=AB**0.5
```

```
      XX=(0.0-P+BB)/2.0
      YY=(0.0-P-BB)/2.0
      PUNCH11,XX,YY
 11   FORMAT(1H 2E18.8)
110   N=N-2
      IF(N-3)17,12,12
 12   B(1)=1.0
      B(2)=A(2)-P
      A(2)=B(2)
      DO13I=3,N,1
      B(I)=A(I)-P*B(I-1)-G*B(I-2)
 13   A(I)=B(I)
      IF(N-3)14,14,5
 14   P=A(2)
      G=A(3)
      GOTO9
 15   T=ABSF(AB)
      BB=SQRTF(T)
      IF(BB-.10E-10)31,32,32
 31   PUNCH33
 33   FORMAT(9H IMAG RTS)
      GOTO100
 32   COMP=(0.0-P)/2.0
150   PUNCH16,COMP
 16   FORMAT(1H E18.8)
      GOTO110
 17   IF(N-2)100,18,100
 18   COMP=0.0-A(2)+P
      GOTO150
 19   P=P+R
      G=G+R
      IF(P-100.0)6,20,20
 20   PUNCH 21
 21   FORMAT(15H P IS OVER 100.)
      R=-0.5
      P=-0.5
      G=-0.5
      GOTO6
      END
```

```
..I    PROGRAM D
..I    PROJECT 923124  UNSYMMETRICAL BEAM
..I    EVALUATION OF BOUNDARY CONDITION DETERMINANT
..LOAD FORTRAN EXECUTE
       DIMENSION RT(5),TOP(5),DEN(5),APHI(5),GAM(5)
       DIMENSION B(11,11),XM(11),DET(11)
       READ2,HA,EI,ZI,HCI,SI,ECZ,ECY,HC,EY,G,DN,DS
   02  FORMAT(1H 7F10.5/1H 4E15.5,F10.3)
       N=10
       AB=EY*ZI/DN/HA
       AD=EY*EI/DN/HA
  100  READ4,S,F,RT(1),RT(2),RT(3),RT(4),RT(5)
   04  FORMAT(1H 2F10.1/1H 2E18.8/1H 2E18.8/1H E18.8)
       AC=S/DN/HA
       DO 35 I=1,5
       TOP(I)=AB*RT(I)*RT(I)+AC*RT(I)-F
       DEN(I)=ECZ*(AC*RT(I)-F)
       IF(DEN(I))10,100,10
   10  APHI(I)=TOP(I)/DEN(I)
       GAM(I)=AD*RT(I)*RT(I)+AC*RT(I)-F
       IF(GAM(I))12,100,12
   12  GAM(I)=TOP(I)*ECY/(GAM(I)*ECZ)
       IF(RT(I))15,14,16
   14  DET(N)=666.6
       GO TO 95
   15  RT(I)=-RT(I)
       RT(I)=RT(I)**.5
       B(1,2*I-1)=1.
       B(1,2*I)=0.
       B(2,2*I-1)=COSF(RT(I)*DS)
       B(2,2*I)=SINF(RT(I)*DS)
       B(3,2*I-1)=GAM(I)
       B(3,2*I)=0.
       B(4,2*I-1)=GAM(I)*B(2,2*I-1)
       B(4,2*I)=GAM(I)*B(2,2*I)
       B(5,2*I-1)=APHI(I)
       B(5,2*I)=0.
       B(6,2*I-1)=APHI(I)*B(2,2*I-1)
       B(6,2*I)=APHI(I)*B(2,2*I)
       B(7,2*I-1)=0.
       B(7,2*I)=RT(I)
       B(8,2*I-1)=-RT(I)*B(2,2*I)
       B(8,2*I)=RT(I)*B(2,2*I-1)
       B(9,2*I-1)=0.
       B(9,2*I)=RT(I)*GAM(I)
       B(10,2*I-1)=-GAM(I)*RT(I)*B(2,2*I)
       B(10,2*I)=GAM(I)*RT(I)*B(2,2*I-1)
       GO TO 35
   16  RT(I)=RT(I)**.5
       B(1,2*I-1)=1.
       B(1,2*I)=1.
       B(2,2*I-1)=EXPF(DS*RT(I))
       B(2,2*I)=1./B(2,2*I-1)
       B(3,2*I-1)=GAM(I)
       B(3,2*I)=GAM(I)
       B(4,2*I-1)=GAM(I)*B(2,2*I-1)
```

```
      B(4,2*I)=GAM(I)*B(2,2*I)
      B(5,2*I-1)=APHI(I)
      B(5,2*I)=APHI(I)
      B(6,2*I-1)=APHI(I)*B(2,2*I-1)
      B(6,2*I)=APHI(I)*B(2,2*I)
      B(7,2*I-1)=RT(I)
      B(7,2*I)=-RT(I)
      B(8,2*I-1)=RT(I)*B(2,2*I-1)
      B(8,2*I)=-RT(I)*B(2,2*I)
      B(9,2*I-1)=GAM(I)*RT(I)
      B(9,2*I)=-B(9,2*I-1)
      B(10,2*I-1)=GAM(I)*RT(I)*B(2,2*I-1)
      B(10,2*I)=-GAM(I)*RT(I)*B(2,2*I)
   35 CONTINUE
      N2=N-2
      DO90 K=1,N2
      K1=K+1
      K2=K+2
      LARG=K1
      VLARG=B(K,K1)
      VLARG=ABSF(VLARG)
      DO63 J=K2,N
      AJK=B(J,K)
      IF(VLARG-ABSF(AJK))64,63,63
   64 VLARG=ABSF(AJK)
      LARG=J
   63 CONTINUE
      IF(LARG-K1)70,70,67
   67 DO68 I=K,N
      TEMP=B(I,K1)
      B(I,K1)=B(I,LARG)
   68 B(I,LARG)=TEMP
   70 IF(B(K,K1))71,100,71
   71 RECIP=1./B(K,K1)
      DO75 J=K2,N
   75 XM(J)=-B(K,J)*RECIP
      DO80 I=K,N
      DO80J=K2,N
   80 B(I,J)=B(I,J)+XM(J)*B(I,K1)
      IF(LARG-K1)85,85,81
   81 DO82 J=1,N
      TEMP=B(K1,J)
      B(K1,J)=B(LARG,J)
   82 B(LARG,J)=TEMP
   85 DO90 J=1,N
      SUM=0.
      DO88 L=K2,N
   88 SUM=SUM+XM(L)*B(L,J)
   90 B(K1,J)=B(K1,J)-SUM
      DET(1)=B(1,1)
      DO120 M=2,N
      DETM=B(M,1)
      DO  110 I=2,M
  110 DETM=B(M,I)*DET(I-1)-DETM*B(I-1,I)
  120 DET(M)=DETM
   95 PUNCH96,S,F,DET(N)
```

```
96   FORMAT(1H 2F10.1,E15.5)
     GO TO 100
     END
```

```
..I   INPUT DATA FOR SYMMETRICAL BEAMS

      BEAM NO. 1

         UNCOUPLED FREQUENCIES
  .0324        .36          90.    29500000.    .000734
     4000.       3000.
     4000.       4000.
     4000.       4100.
     4000.       5000.
     4000.     122000.
       .          .
       .          .

         COUPLED FREQUENCIES

  .1276      .16         .2520       .36         90.        .5055
  .001953      29500000.  11500000.    .000734
     4000.     57000.
     4000.     58000.
     4000.     59000.
     4000.    292000.
     4000.    294000.
       .          .
       .          .


      BEAM NO. 2

         UNCOUPLED FREQUENCIES

  .0802        .5049        90.    29500000.    .000734
    10000.     1000.
    10000.     2000.
    10000.     3000.
    10000.     4000.
    10000.     5000.
       .          .
       .          .

         COUPLED FREQUENCIES

  .002963      29500000.  11500000.    .000734
  .3158      .3960       .6315       .5049       90.        .6829
    10000.     38000.
    10000.     39000.
    10000.     40000.
    10000.     41000.
    10000.     42000.
       .          .
       .          .
       .          .
```

```
..I    INPUT DATA FOR UNSYMMETRICAL BEAMS (3,4)
       INPUT TO CHARACTERISTIC EQUATION

    0.42    0.0457    0.214        0.26        0.43       0.540      0.337
   0.00231     29500000.   11500000.   0.000734      90.
     6000.      2000.
     6000.      4000.
     6000.      6000.
     6000.      8000.
     6000.     10000.
     4000.     10000.
     4000.     12000.
       .          .
       .          .
       .          .
```